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Rationality conjecture for finite CW-complexes with unique spherical cohomology class and high-torsion lens spaces

K. Boutahir, A. El Houari, Y. Rami [@]

Department of Mathematics, My Ismail University, B.P. 11201 *Zitoune, Meknès, Morocco*

k.boutahir@umi.ac.ma, *a.elhouari.edu@gmail.com*, *y.rami@umi.ac.ma*

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Abstract: This paper examines the weak Rationality Conjecture for $(r-1)$-connected CW-complexes X admitting a unique spherical cohomology class $u \in H^r(X,\mathbb{Z})$ for some $r \geq 2$. We affirm this conjecture for such complexes of dimension kr where $u^k \neq 0$. We provide a non-exhaustive list of finite CW-complexes satisfying the conjecture. Furthermore, we investigate some high-torsion Lens spaces L_m^{2n+1}, which are defined as orbit spaces $S^{2n+1}/\mathbb{Z}_m$. The generating functions of these lens spaces confirm both the Weak and Strong Rationality Conjectures under certain conditions.

Key words: (Higher) Topological complexity, Hopf invariant, Zero-divisor-cup-length, spherical cohomology class.

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1. Introduction and statement of results

Topological complexity $TC(X)$, introduced by M. Farber, and its higher analogue $TC_n(X)$ $(n \geq 2)$ (also known as sectional topological complexity), introduced by Y. Rudyak, are numerical homotopy invariants that establish a powerful connection between topology and robotics. They measure the complexity of motion planning algorithms for a physical system whose configuration space is X, with $TC_2(X) = TC(X)$.

For a topological space X and $n \geq 2$, consider the fibration

$$\begin{array}{rcl} \pi_n : X^I & \longrightarrow & X^n \\ \gamma & \longmapsto & \left(\gamma(0), \gamma\left(\frac{1}{n}\right), \ldots, \gamma\left(\frac{n-1}{n}\right), \gamma(1)\right). \end{array}$$

This fibration serves as a substitute for the iterated diagonal map $\Delta_n : X \to X^n$, from which we define $TC_n(X) = \mathrm{secat}(\pi_n)$. In other words, $TC_n(X)$ is the smallest integer $k \geq 0$ such that there exists an open cover

[@] Corresponding author

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$X^n = U_0 \cup \cdots \cup U_k$ with the property that each set U_i $(0 \leq i \leq k)$ admits a homotopy section $s_i : U_i \to X^I$, i.e., $\pi_n \circ s_i = J_i : U_i \hookrightarrow X^n$.

Computing $TC_n(X)$ $(n \geq 2)$ is challenging, so it is often approximated using algebraic invariants like the *zero-divisor-cup-length*:

$$\mathrm{zcl}_n(X, \mathbb{K}) = \max \left\{ j \geq 1 : \begin{array}{c} \exists\, a_1, \ldots, a_j \in \mathrm{Ker}(\smile_{n,\mathbb{K}}) \\ \text{such that } a_1 \cdot \ldots \cdot a_j \neq 0 \end{array} \right\}, \tag{1.1}$$

where $\smile_{n,\mathbb{K}}$: $H^*(X, \mathbb{K})^{\otimes n} \to H^*(X, \mathbb{K})$ defines a multiplication in the graded cohomology algebra, generalizing the standard *cup product* (see Section 2).

Considering all the numbers $TC_n(X)$, we obtain the formal power series

$$\mathcal{F}_X(x) = \sum_{n=1}^{n=\infty} TC_{n+1}(X) x^n \tag{1.2}$$

called the *TC-generating function*.

In this paper, we consider the following conjectures posed, respectively, in [8] and [7]:

Conjecture 1.1. (Strong Rationality Conjecture) For any finite CW-complex X, the TC-generating function $\mathcal{F}_X(x)$ is a rational function with a single pole of order 2 at $x = 1$. That is:

$$\mathcal{F}_X(x) = \frac{P_X(x)}{(1-x)^2},$$

where $P_X(x)$ is an integral polynomial satisfying $P_X(1) = \mathrm{cat}(X)$.

Conjecture 1.2. (Weak Rationality Conjecture) For any finite CW-complex X, the TC-generating function $\mathcal{F}_X(x)$ is a rational function with a single pole of order 2 at $x = 1$. That is:

$$\mathcal{F}_X(x) = \frac{P_X(x)}{(1-x)^2},$$

where $P_X(x)$ is an integral polynomial.

The study of Rationality Conjectures began with Farber and Oprea [8], who observed a remarkable consistency in the topological complexity of configuration spaces of F-type groups. They conjectured that the generating function is rational with a pole of order 2 at $x = 1$. Moreover, the Strong

Rationality Conjecture claims that the numerator must satisfy the condition $P_X(1) = \mathrm{cat}(X)$ [8], They specifically inquire if this conjecture holds true for the significantly important category of hyperbolic groups. Later, Farber, Kishimoto and Stanley [7] developed a counterexample showing that the Strong Rationality Conjecture fails in general: the predicted connection to the Lusternik-Schnirelmann category does not always hold. This motivated the formulation of the Weak Rationality Conjecture, which only requires the rational form of the generating function without the category condition.

Aguilar-Guzman et al. [1] does not focus on proving the conjecture outright. Instead, they deepen its understanding, providing intricate examples through right-angled Artin groups.

Hughes and Li [11] answered affirmatively a question of Farber and Oprea [8], proving that the Rationality Conjecture holds for torsion-free hyperbolic groups. Consequently, this paper signifies a substantial advancement and a noteworthy affirmation of the rationality conjecture within the realm of groups, validating it for one of the most extensively examined categories of groups in contemporary geometry.

Although the original conjecture is false in general, it has been established for a broad class of spaces, confirming the underlying intuition.

In [7, Theorem 1] Farber et al. gave a sufficient condition involving the series $\mathrm{zcl}_n(X;\mathbb{K})$ that affirms Conjecture 1.2 as follows:

THEOREM 1.3. *Let X be a connected finite CW-complex such that, for any integer n and for some field $\mathbb{K}$, $TC_n(X) = \mathrm{zcl}_n(X;\mathbb{K})$. Under this condition, Conjecture 1.2 holds.*

This research examines two categories:

- finite $(r-1)$-connected CW-complexes X with $H^r(X,\mathbb{Z}) \cong \mathbb{Z}u$, $r \geq 2$, and $\dim X = kr$ where $u^k \neq 0$,
- some high-torsion Lens spaces L_m^{2n+1}, which are defined as orbit spaces of $S^{2n+1} = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1\}$ under $\mathbb{Z}_m = \{z \in \mathbb{C} \mid z^m = 1\}$ [10].

For the first category, we support the Weak Rationality Conjecture, broadening findings to spaces with cohomology $H^*(X,\mathbb{K}) \cong \mathbb{K}[u]/(u^{k+1})$ for $r = 2$ or 4, and kr-skeletons of $\mathbb{S}^r_\infty$ [9, 16, 12]. Regarding Lens spaces, Daundkar's findings indicate $TC_k(L_m^{2n+1}) = k \cdot (2n+1)$ under certain conditions, thus confirming both conjectures.

This paper aims to:

1. Prove the Weak Rationality Conjecture for $(r-1)$-connected CW-complexes with $H^r(X,\mathbb{Z}) \cong \mathbb{Z}u$, $r \geq 2$, and $\dim X = kr$ where $u^k \neq 0$.
2. Confirm both Rationality Conjectures for certain high-torsion Lens spaces L_m^{2n+1}.

In what follows, we consider a finite $(r-1)$-connected CW-complex X with $r \geq 2$, and we seek sufficient conditions for $\mathrm{zcl}_n(X;\mathbb{K}) = TC_n(X)$ for all $n \geq 2$ inspired by Theorem 1.3. To this end, we assume that:

$$H^r(X,\mathbb{Z}) \cong \mathbb{Z}u, \qquad \text{with } u^k \neq 0 \text{ for some } k \geq 2. \tag{1.3}$$

Notice that we do not require that k is the greatest integer satisfying (1.3). The case where $u^2 = 0$ corresponds to $X = \mathbb{S}$ which by [7] verifies Conjecture 1.2. The assumption $k \geq 2$ requires implicitly assuming that r is even.

The "unique spherical cohomology class" hypothesis means that $\mathbb{Z}u \cong H^r(X,\mathbb{Z})$ where u is spherical. This condition is natural because it ensures the cohomology ring is generated by a single element coming from the Hurewicz image of the homotopy group. Key examples include skeleta of James' reduced product space $\mathbb{S}^r_\infty$, where the attaching maps are chosen so that the top-dimensional cohomology class remains non-zero under cup products.

To provide a topological interpretation of our results, we recall that, for any respective generating classes $a \in H_r(X,\mathbb{Z}) \cong \mathbb{Z}$ and $[f] \in \pi_r(X) \cong \mathbb{Z}$, we have $< f^*(u), a > = < u, f_*(a) > \neq 0$ and consequently, $f^*(u) \neq 0$. Any class $u \in H^r(X,\mathbb{Z})$ satisfying this property is called a *spherical cohomology class* [9, Definition 1.2 and Lemma 3.1].

Our main result reads as follows

Theorem 1.4. *Any finite $(r-1)$-connected kr-dimensional CW-complex that satisfies condition (1.3) also satisfies the Conjecture 1.2.*

Let X be a simply connected space such that $H^*(X,\mathbb{K}) \cong \mathbb{K}[u]/(u^{k+1})$ $(k \geq 3)$. Referring to [16, Theorem 2] and [10, Theorem 4L 10] we know that $r = |u|$ should be either 2 or 4. Clearly X satisfies (1.3), hence we in fact obtain the following

Corollary 1.5. *Let X be an $(r-1)$-connected kr-dimensional CW-complex whose cohomology over $\mathbb{Z}$ is a truncated polynomial algebra. Then X satisfies the Conjecture 1.2.*

Another consequence of Theorem 1.4 concerns kr-skeletons of the reduced product complex $\mathbb{S}^r_\infty$ of the sphere $\mathbb{S}^r$. The latter was introduced by James in

[12] to extend the well known Hopf invariant to such spaces. We obtain the following:

COROLLARY 1.6. *Every kr-skeleton* $\mathbb{S}^r_k =: \mathbb{S}^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \cdots \cup_{\beta_k} e^{kr}$ *of* $\mathbb{S}^r_\infty$ *satisfies Conjecture* 1.2.

In the remainder of this work, we devote Section 2 to the characterization of fields $\mathbb{K}$ for which $u^k_{\mathbb{K}} \neq 0$, Section 3 to the proof of Theorem 1.4 and Section 4 to Lens spaces .

2. Fields satisfying $u^k_{\mathbb{K}} \neq 0$

Let X be a CW-complex, the cup-product may be defined in terms of the Alexander-Whitney diagonal approximation:

$$\smile : H^p(X, G_1) \otimes H^q(X, G_2) \longrightarrow H^{p+q}(X, G_1 \otimes G_2) \tag{2.1}$$

given, in general, for any groups G_1 and G_2 by:

$$(a \smile b)(\sigma) = a(\sigma_{|[v_0, v_1, \ldots, v_p]}) \otimes b(\sigma_{|[v_p, v_{p+1}, \ldots, v_{p+q}]}).$$

We consider in (2.1) $G_1 = \mathbb{Z}$, $G_2 = \mathbb{K}$ where $\mathbb{K}$ is an arbitrary field and $u \in H^p(X, \mathbb{Z})$. Since $1_{\mathbb{K}} \in H^0(X, \mathbb{K})$ and $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{K} \cong \mathbb{K}$ we have $u_{\mathbb{K}} := u \smile 1_{\mathbb{K}} \in H^p(X, \mathbb{K})$, therefore, we make the identification $u^k_{\mathbb{K}} := u^k \smile 1_{\mathbb{K}}$ which allows us to extend the *zero-divisor-cup-length* invariant given in (1.1) to $H^*(X, \mathbb{Z})$.

In all that follows, unless otherwise stated, X will denote an $(r-1)$-connected finite CW-complex satisfying condition (1.3). It follows that $H^{kr}(X, \mathbb{Z})$ has a finitely generated abelian group structure given by the identification:

$$H^{kr}(X, \mathbb{Z}) = \mathbb{Z}^{m_k} \oplus (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \oplus (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/p_{\theta(k)}^{\alpha_{\theta(k)}}\mathbb{Z}). \tag{2.2}$$

We will also consider in addition to $\mathcal{P}$, the set of all prime numbers, the families:

$$\mathcal{P}_{u^k} := \{p_1, p_2, \ldots, p_{\theta(k)}\}, \tag{2.3}$$

$$\mathcal{Q}_{u^k} := \{q_1, q_2, \ldots, q_{\gamma(k)} \mid s.t.\ u^k = q_1^{f_1} q_2^{f_2} \ldots q_{\gamma(k)}^{f_{\gamma(k)}} \omega;\ \omega \in \mathbb{Z}^{m_k}\}. \tag{2.4}$$

Here ω is an element of the free part $\mathbb{Z}^{m_k}$ of $H^{kr}(X, \mathbb{Z})$.

Although $u \in H^r(X,\mathbb{Z})$ has infinite additive order, there may be an $m \in \mathbb{N}^*$ such that $mu^{l+1} = 0$ for some $1 \leq l \leq k-1$ (i.e., at the cochain level, $mu^{l+1} = \delta v$ for some $v \in C^{(l+1)r-1}(X)$). We consider two cases based on the additive order $o(u^k)$ of u^k. We denote by $\mathrm{Free}(G)$, the free part of a group G.

(*) $o(u^k)$ is infinite, that is $\mathrm{Free}(H^{lr}(X,\mathbb{Z})) \neq 0$ for each $1 \leq l \leq k$. Then (1.3) implies that $u^k_{\mathbb{K}} \neq 0$ for any field $\mathbb{K}$ whose characteristic is zero or is in $\mathcal{P}\backslash\mathcal{Q}_{u^k}$.

(**) $o(u^k)$ is finite, that is, $mu^{l+1} = 0$ for some $m \in \mathbb{N}^*$ and every $1 \leq l \leq k-1$. In particular, $o(u^k) =: p_1^{\beta_1} p_2^{\beta_2} \cdots p_{\theta(k)}^{\beta_{\theta(k)}}$ $(1 \leq i \leq \theta(k))$ divides m with $0 \leq \beta_i \leq \alpha_i$ and at least one $\beta_i > 0$.

Since $\mathcal{Q}_{u^l}$ is defined similarly to $\mathcal{Q}_{u^k}$, condition (1.3) implies that $u^l_{\mathbb{K}} \neq 0$ $(1 \leq l \leq k-1)$ for every field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \in \mathcal{P}\backslash\mathcal{Q}_{u^l}$. On the other hand, for any field $\mathbb{K}$ whose $\mathrm{char}(\mathbb{K}) \in \mathcal{P}\backslash\mathcal{P}_{u^k}$, we get $u^{l+1}_{\mathbb{K}} \neq 0, \ldots, u^k_{\mathbb{K}} \neq 0$.

We then consider:

$$\mathcal{P}(i) = \{0\} \cup \mathcal{P}\backslash\mathcal{Q}_{u^k} \qquad \text{and} \qquad \mathcal{P}(ii) = \mathcal{P}\backslash(\mathcal{P}_{u^k} \cup \mathcal{Q}_{u^l}). \tag{2.5}$$

In summary, the unique spherical cohomology class u and the integer k involved in condition (1.3) satisfy the property:

Proposition 2.1. (1) *If $o(u^k)$ is infinite, then $u^k_{\mathbb{K}} \neq 0$ if and only if* $\mathrm{char}(\mathbb{K}) \in \mathcal{P}(i)$.

(2) *If $o(u^k)$ is finite, then $u^k_{\mathbb{K}} \neq 0$ if and only if* $\mathrm{char}(\mathbb{K}) \in \mathcal{P}(ii)$.

In summary: when u^k has infinite order, we avoid primes dividing the torsion part of lower powers; when u^k has finite order, we additionally exclude primes dividing $o(u^k)$ itself. For example, when $X = \mathbb{S}^r_k$ (infinite order), any characteristic works; whereas if u^2 has order 2, characteristic 2 must be avoided.

3. Proof of Theorem 1.4

In the previous section we specified classes of fields satisfying the condition $u^k_{\mathbb{K}} \neq 0$. In this section, we refine these classes to identify fields $\mathbb{K}$ for which the condition $TC_n(X) = \mathrm{zcl}_n(X,\mathbb{K})$ holds for all integers $n \geq 2$. To do so, consider (for any $n \geq 3$) in $H^n(X,\mathbb{K})$ the following sequence:

$$A_1 = u_{\mathbb{K}} \otimes 1 \otimes \cdots \otimes 1\,, \qquad A_2 = 1 \otimes u_{\mathbb{K}} \otimes 1 \otimes \cdots \otimes 1\,, \\ \ldots\,, \qquad A_n = 1 \otimes \cdots \otimes 1 \otimes u_{\mathbb{K}}.$$

This is special in the sense that differences $A_i - A_j$ $(1 \leq i \neq j \leq n)$ obviously generate the ideal $\mathrm{Ker}(\smile_{n,\mathbb{K}})$. Next, we consider the following products:

$$\xi_{(n,k)} = \left[\left(\prod_{i=2}^{i=n}(A_1 - A_i)\right)(A_2 - A_3)\right]^k \tag{3.1}$$

and

$$\begin{aligned} \mu_{(n,k)} &= \xi_{(n,k-1)}[(A_1 - A_2)(A_1 - A_3)\cdots(A_1 - A_n)] \\ &= \xi_{(n,k)}/(A_2 - A_3). \end{aligned} \tag{3.2}$$

The following lemma involves the integer:

$$\lambda_{(3,k)} = \sum_{0 \leq i \leq k} (-1)^i (C_k^i)^3. \tag{3.3}$$

It is the main technical tool we will use to prove our results.

Lemma 3.1. *Let* $\mathrm{char}(\mathbb{K}) \in \mathcal{P}(i)$ *(resp.* $\mathcal{P}(ii)$*). Then:*

(a) $\dim \mathrm{Span}\left\{u_{\mathbb{K}}, u_{\mathbb{K}}^2, \ldots, u_{\mathbb{K}}^k\right\} = k$*;*

(b) *for any* $n \geq 3$*:*

(i) $\xi_{(n,k)} = \lambda_{(3,k)} u_{\mathbb{K}}^k \otimes \cdots \otimes u_{\mathbb{K}}^k$ *when* k *is even and* $\xi_{(n,k)} = 0$ *when* k *is odd.*

(ii) $\mu_{(n,k)}(A_1 - A_n) = 2(-1)^{n-1}\lambda_{(3,k-1)} u_{\mathbb{K}}^k \otimes \cdots \otimes u_{\mathbb{K}}^k$ *when* k *is odd.*

Proof. Clearly the first assertion is evident from the cases $(*)$ and $(**)$. For the second, notice first the following inductive formulas for any $n \geq 3$, and $l \geq 1$:

$$\xi_{(n+1,k)} = \xi_{(n,k)}(A_1 - A_{n+1})^k \qquad \text{and} \qquad \xi_{(n,k+l)} = \xi_{(n,k)}\xi_{(n,l)}.$$

In both cases $(*)$ and $(**)$ above, we have $u_{\mathbb{K}}^j = 0, \forall j > k$, hence, a straightforward computation gives

$$\xi_{(n,k)} = \lambda_{(n,k)} u_{\mathbb{K}}^k \otimes \cdots \otimes u_{\mathbb{K}}^k \tag{3.4}$$

(some constant $\lambda_{(n,k)} \in \mathbb{Z}$). It remains to determine $\lambda_{(n,k)}$. Using once more $u^j_{\mathbb{K}} = 0$ for all $j > k$, we deduce that in the relation $\xi_{(n+1,k)} = \xi_{(n,k)}(A_1 - A_{n+1})^k$, the only term to be retained from Newton's formula

$$(A_1 - A_{n+1})^k = \sum_{i=0}^{i=k} (-1)^{k-i} C^i_k A^i_1 A^{k-i}_{n+1}$$

is $(-1)^k A^k_{n+1}$. Thus,

$$\xi_{(n+1,k)} = (-1)^k \xi_{(n,k)} A^k_{n+1} = (-1)^k \lambda_{(n,k)} u^k_{\mathbb{K}} \otimes \cdots \otimes u^k_{\mathbb{K}}, \quad n+1 \text{ factors.}$$

Therefore $\lambda_{(n+1,k)} = (-1)^k \lambda_{(n,k)} = \cdots = (-1)^{(n-2)k} \lambda_{(3,k)}$. Consequently, we obtain:

$$\lambda_{(n,k)} = (-1)^{(n-3)k} \lambda_{(3,k)}, \qquad \forall\, n \geq 3\,. \tag{3.5}$$

Next, using Newton's formula in the product

$$\xi_{(3,k)} = (-1)^k (A_1 - A_2)^k (A_1 - A_3)^k (A_2 - A_3)^k,$$

and the fact that $u^j_{\mathbb{K}} = 0$ for all $j > k$, we obtain the unexpectedly integer:

$$\lambda_{(3,k)} = \sum_{0 \leq i \leq k} (-1)^i (C^i_k)^3.$$

But, when k is odd, since $C^i_k = C^{k-i}_k$ $(1 \leq i \leq k)$, we have $\lambda_{(3,k)} = 0$. Thus, Using (3.4) we then establish (i).

Next, consider $\mu_{(n,k)} = \xi_{(n,k-1)}[(A_1 - A_2)(A_1 - A_3) \cdots (A_1 - A_n)]$. Formulas relating coefficients and roots of $Q(A_1) = (A_1 - A_2)(A_1 - A_3) \cdots (A_1 - A_n)$, combined with constraints $u^j_{\mathbb{K}} = 0$, for all $j > k$, lead us to consider only the constant coefficient $q_0 = (-1)^{n-1} A_2 A_3 \cdots A_n$ and the coefficient $q_1 = (-1)^{n-2} \sum_{2 \leq i_1 < \ldots < i_{n-2} \leq n} A_{i_1} \cdots A_{i_{n-2}}$ of A_1. Therefore,

$$\begin{aligned}\mu_{(n,k)} = (-1)^{n-1} \lambda_{(n,k-1)} \Big[& u^{k-1}_{\mathbb{K}} \otimes u^k_{\mathbb{K}} \cdots \otimes u^k_{\mathbb{K}} \\ & - \sum_{j=2}^{j=n} u^k_{\mathbb{K}} \otimes \cdots \otimes u^k_{\mathbb{K}} \otimes u^{k-1}_{\mathbb{K}} \otimes u^k_{\mathbb{K}} \cdots \otimes u^k_{\mathbb{K}} \Big]\end{aligned}$$

with $u^{k-1}_{\mathbb{K}}$ in the $j-th$ place. It follows immediately from the above that, when k is odd,

$$\mu_{(n,k)}(A_1 - A_n) = 2(-1)^{n-1} \lambda_{(3,k-1)} u^k_{\mathbb{K}} \otimes \cdots \otimes u^k_{\mathbb{K}}. \tag{3.6}$$

Then (ii) is also satisfied. ∎

We recall the following fundamental theorem, which will be used below. It gives (effective) lower and upper bounds of the higher topological complexities [2].

THEOREM 3.2. *For any s-connected CW-complex X ($s \geq 1$), any field $\mathbb{K}$ and any integer $n \geq 2$ we have*

$$\mathrm{zcl}_n(X, \mathbb{K}) \leq TC_n(X) \leq \frac{n \dim X}{s+1}.$$

Hereafter, $\mathbb{K}$ is assumed to satisfy, according to the case under consideration, the property 2.1 (1) or (2).

Proof of Theorem 1.4. Using Theorem 3.2 with $s = r-1$ and $\dim X = kr$ as in the hypothesis we get $TC_n(X) \leq nk$ for every $n \geq 2$. If $n = 2$ one shows easily (for any $\mathbb{K}$) that $\xi_{(2,k)} = (A_1 - A_2)^{2k} \neq 0$, thus, $2k \leq \mathrm{zcl}_2(X, \mathbb{K})$ and once more by Theorem 3.2 we obtain

$$TC_2(X) = \mathrm{zcl}_2(X, \mathbb{K}) = 2k.$$

If $n \geq 3$, from Lemma 3.1 and formulas (3.1) and (3.4) (resp. (3.2) and (3.6)) we see that $nk \leq \mathrm{zcl}_n(X, \mathbb{K})$ if and only if $\mathrm{char}(\mathbb{K})$ is coprime with $\lambda_{(3,2k)}$ if k is even (resp. with $2\lambda_{(3,k-1)}$ if k is odd). Then, it suffices to use Dirichlet's prime number theorem which ensures existence of at least one field $\mathbb{K}$ whose characteristic satisfies the first (resp. the second) condition. Note that Theorem 1.3 only requires the existence of some field $\mathbb{K}$ for which $TC_n(X) = \mathrm{zcl}_n(X, \mathbb{K})$ holds for all $n \geq 2$; we do not need uniformity across all characteristics, Dirichlet's theorem guarantees at least one prime p coprime to the given integer, which suffices for our application. Therefore, $TC_n(X) = \mathrm{zcl}_n(X, \mathbb{K})$ for every $n \geq 2$. To conclude, we apply Theorem 1.3. ∎

Definitely, the integers $\lambda_{(3,k)}$, for any even integer k, play an important role in the affirmation or non-affirmation of the Conjecture 1.2. *Table* 1 below, presents all prime factors in the factorization of $\lambda_{(3,k)}$ when k varies from 2 to 40. It is easy to show that 2 and 3 divide each $\lambda_{(3,k)}$. But, while a meticulous analysis shows that prime numbers $p \geq 5$ appear and disappear depending on the value of k, we have not yet found a proof of this. We leave this as an open question.

Table 1: Prime factors of $\lambda_{(3,k)}$ for even k from 2 to 40

k	prime factors of $\lambda_{(3,k)}$
2	2, 3
4	2, 3, 5
6	2, 3, 5, 7
8	2, 3, 5, 7, 11
10	2, 3, 7, 11, 13
12	2, 3, 7, 11, 13, 17
14	2, 3, 5, 11, 13, 17, 19
16	2, 3, 5, 11, 13, 17, 19, 23
18	2, 3, 5, 11, 13, 17, 19, 23
20	2, 3, 5, 7, 11, 13, 17, 19, 23, 29
22	2, 3, 5, 7, 13, 17, 19, 23, 29, 31
24	2, 3, 5, 7, 13, 17, 19, 23, 29, 31
26	2, 3, 5, 7, 17, 19, 23, 29, 31, 37
28	2, 3, 5, 17, 19, 23, 29, 31, 37, 41
30	2, 3, 5, 11, 17, 19, 23, 29, 31, 37, 41, 43
32	2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47
34	2, 3, 5, 7, 11, 19, 23, 29, 31, 37, 41, 43, 47
36	2, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, 41, 43, 47, 53
38	2, 3, 5, 7, 11, 13, 23, 29, 31, 37, 41, 43, 47, 53
40	2, 3, 5, 7, 11, 13, 23, 29, 31, 37, 41, 43, 47, 53, 59

The integers $\lambda_{(3,k)} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}^3$ occurring in Lemma 3.1 are known as *alternating Franel numbers*. Although the classical Franel numbers $f_k = \sum_{i=0}^{k}\binom{k}{i}^3$ have a three-term recurrence relation with polynomial coefficients [15], their alternating versions have a much simpler form: they are zero for odd k, and for even $k = 2m$ they are explicitly expressed as [14, Lemma 2.11]

$$\lambda_{(3,2m)} = (-1)^m\binom{3m}{m,m,m} = (-1)^m\frac{(3m)!}{(m!)^3}, \tag{3.7}$$

i.e., signed trinomial coefficients. This closed form expression is a consequence of *Dickson's identity* [6], which can be proven algorithmically using the Zeilberger-Wilf method [14, Proof of Lemma 2.11]. The prime factorizations in Table 1 imply which of the characteristics are to be excluded; in particular, 2 and 3 divide every non-zero $\lambda_{(3,k)}$, while larger primes appear sporadically.

3.1. Examples Recall from [3] that the reduced product complex $\mathbb{S}^r_\infty$ of the sphere $\mathbb{S}^r$ introduced by James in [12] is homotopy equivalent to $\Omega\mathbb{S}^{r+1}$. It has a natural decomposition

$$\mathbb{S}^r_\infty = \mathbb{S}^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \cdots \cup_{\beta_k} e^{kr} \ \cup \cdots .$$

The kr-skeleton

$$\mathbb{S}^r_k =: \mathbb{S}^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \cdots \cup_{\beta_k} e^{kr} = \mathbb{S}^r_{k-1} \cup_{\beta_r} E^{kr}$$

(E^{kr} being the kr-cell of dimension kr) of $\mathbb{S}^r_\infty$ is used by James [13] to generalize Steenrod's definition of the classical Hopf invariant

$$h_2 : \pi_{2r-1}(\mathbb{S}^r) \longrightarrow \mathbb{Z}$$

as follows: Let $[\alpha] \in \pi_{kr-1}(\mathbb{S}^r_{k-1})$ and choose $a_1,\ a_{(k-1)r}$ and x generators of dimension r, $(k-1)r$ and kr in the cohomology ring $H^*(\mathbb{S}^r_{k-1} \cup_{\beta_r} E^{kr})$, then the *generalized Hopf invariant* is defined by the relation via Poincaré duality property as follows:

$$\begin{array}{rcl} h^r_k : \pi_{kr-1}(\mathbb{S}^r_{k-1}) & \longrightarrow & \mathbb{Z} \\ \alpha & \longmapsto & h^r_k(\alpha) \end{array}$$

such that $a_1 \cup a_{(k-1)r} = h^r_k(\alpha)x$.

The following theorem is paramount to select examples of the form $\mathbb{S}^r_k$ satisfying the Conjecture 1.2:

Theorem 3.3. ([3, Theorem A])

$$Im(h^r_k) = \begin{cases} \mathbb{Z} & \text{if } r \in \{2,4,8\} \text{ and } k = 2\,, \\ \mathbb{Z} & \text{if } r = 2 \text{ and } k \text{ is a prime number}\,, \\ k\mathbb{Z} & \text{otherwise.} \end{cases}$$

Based on this theorem, we have the following

Corollary 3.4. *Every kr-skeleton $\mathbb{S}^r_k =: \mathbb{S}^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \cdots \cup_{\beta_k} e^{kr}$ of $\mathbb{S}^r_\infty$ verifies Conjecture* 1.2.

Proof. Let a_{ir} $(2 \leq i \leq k)$ denote the top cohomology class of $\mathbb{S}^r_i = \mathbb{S}^r \cup_{\beta_2} e^{2r} \cup_{\beta_3} \cdots \cup_{\beta_i} e^{ir}$ and by a_1 that of $\mathbb{S}^r$. Since each $h^r_i(\alpha)$ comes from duality property of $H^*(\mathbb{S}^r_{k-1} \cup_{\beta_r} E^{kr})$, it is non zero. An easy induction shows that $a_1^k = h^r_2(\beta_2)h^r_3(\beta_3)\cdots h^r_{k-1}(\beta_k)a_{rk}$. It follows that $\mathbb{S}^r_k$ satisfies condition (1.3), hence, being of dimension kr it also verifies Conjecture 1.2. ∎

Examples 3.5. The following complexes satisfy Conjecture 1.2.

1. $\mathbb{S}^2_2 = \mathbb{S}^2 \cup_{\beta_2} e^4$, i.e. $r = k = 2$, where $\beta_2 : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map, that is, $h_2(\beta_2) = 1$ hence $x = a_1 \cup a_1$. It results that $H^*(\mathbb{S}^2_2, \mathbb{Z}) \cong \mathbb{Z}[a_1]\backslash(a_1^3)$ so that, by Corollary 1.6, $\mathbb{S}^2_2$ verifies Conjecture 1.2.

2. $\mathbb{S}^2_3 = \mathbb{S}^2 \cup_{\beta_2} e^4 \cup_{\beta_3} e^6$, i.e. $r = 2$ and $k = 3$, with $\beta_2 : \mathbb{S}^3 \to \mathbb{S}^2$ as in the previous example and $\beta_3 = [i_2, [i_2, i_2]] : \mathbb{S}^5 \to \mathbb{S}^2 \cup_{\beta_2} e^4$ which satisfies $h_3(\beta_3) = 3$ [3, §3]. Hence, $3x = a_1^3$ so that $a_1^3 \neq 0$ (over $\mathbb{Z}$). It results that $\mathbb{S}^2_3$ satisfies (1.3). Being of dimension $kr = 6$, it then verifies Conjecture 1.2.

3. $\mathbb{S}^4_3 = \mathbb{S}^4 \cup_{\beta_2} e^8 \cup_{\beta_3} e^{12}$, i.e. $r = 4$ and $k = 3$, with $\beta_2 : \mathbb{S}^7 \to \mathbb{S}^4$ the Hopf map and $\beta_3 = [i_4, [i_4, i_4]] : \mathbb{S}^{11} \to \mathbb{S}^4 \cup_{\beta_2} e^8$ satisfying also $h_3(\beta_3) = 3$. Hence, once more $3x = a_1^3 \neq 0$ (over $\mathbb{Z}$). It results that $\mathbb{S}^4_3$ satisfies (1.3). Being of dimension $kr = 12$, it then verifies Conjecture 1.2.

4. $\mathbb{S}^4_4 = \mathbb{S}^4 \cup_{\beta_2} e^8 \cup_{\beta_3} e^{12} \cup_{\beta_4} e^{16}$, i.e. $r = 4$ and $k = 4$, with $\beta_2 : \mathbb{S}^7 \to \mathbb{S}^4$, the Hopf map and $\beta_3 = [i_4, [i_4, i_4]] : \mathbb{S}^{11} \to \mathbb{S}^4 \cup_{\beta_2} e^8$ as in the previous example. Consider $\beta_4 : \mathbb{S}^{15} \to \mathbb{S}^4 \cup_{\beta_2} e^8 \cup_{\beta_3} e^{12}$ satisfying $h^4_4(\beta_4) = 4$ as provided by Theorem 1.4. Hence, $12x = a_1^4 \neq 0$. Once more, since $\dim \mathbb{S}^4_4 = 16$, we conclude that $\mathbb{S}^4_4$ also verifies Conjecture 1.2.

5. $\mathbb{S}^8_8 = \mathbb{S}^8 \cup_{\beta_2} e^{16} \cup_{\beta_3} e^{24} \cup_{\beta_4} e^{32} \cup_{\beta_5} e^{40} \cup_{\beta_6} e^{48} \cup_{\beta_7} e^{56} \cup_{\beta_8} e^{64}$, i.e. $r = 8$ and $k = 8$. We consider β_2, the Hopf map and, following [3], the attaching maps $\beta_i = [i_8]^i$, that is such the $h^8_i(\beta_i) = i$, for every $3 \leq i \leq 8$. Hence, $\frac{8!}{2}x = a_1^8 \neq 0$. Thus by Theorem 3.3, $\mathbb{S}^8_8$ verifies Conjecture 1.2.

6. The examples above correspond to the cases where $r = 2, 4, 8$. Now, by Theorem 3.3, when $r \neq 2, 4, 8$, we may take $\beta_i = [i_r]^i$ $(2 \leq i \leq k)$ to obtain a CW-complex $\mathbb{S}^r_k$ such that the product of its generalized Hopf invariant is $h_2(\beta_2) \cdot \ldots \cdot h_k(\beta_k) = k!$. We then conclude, using Theorem 1.4, that $\mathbb{S}^r_k$ satisfies Conjecture 1.2.

4. Rationality conjectures of certain Lens spaces

Daundkar [5] recently established exact calculations of the topological complexity for high-torsion Lens spaces L_m^{2n+1}. We build upon these results to verify both the Weak and Strong Rationality Conjectures for this class of spaces. Specifically, Daundkar proved that $TC_k(L_m^{2n+1}) = k \cdot (2n+1)$ under certain conditions on the integer m [5, Theorem 5.3 and Theorems 5.5–5.7]. Our contribution is to show that this linear dependence on k implies the rationality of the generating function, thereby confirming the conjectures.

Theorem 4.1. ([5]) *Assume L_m^{2n+1} is a Lens space of dimension $2n+1$ with $m > 1$. Then, the higher topological complexity of L_m^{2n+1} is given by $TC_k(L_m^{2n+1}) = k \cdot (2n+1)$ if one of the following holds.*

(i) *m does not divide $\binom{2n}{n}^{\lfloor k/2 \rfloor}$.*

(ii) *For a prime p, and $n = n_0 + n_1 p + \cdots + n_k p^k$ the p-adic representation of n, where $n_i \in \{0, \ldots, p-1\}$ and $r_i(n) = 0$ if $2n_i < p$ and or $r_i(n) = r$ if $2n_i \geq p$, where $r = \max\{j \mid n_{i+1} = \cdots = n_{i+j-1} = \frac{p-1}{2}\}$, and $p^{\alpha_p(n)\cdot\lfloor k/2 \rfloor + 1}$ divides m, with $\alpha_p(n) = \sum_{i=0}^{k} r_i$.*

(iii) *For a prime $p \geq 3$, $n_i \leq \frac{p-1}{2}$ in the p-adic representation, and $m = p$.*

(iv) *If $m = 2^r$ and $\alpha_2(n)\cdot\lfloor k/2 \rfloor \leq r-1$. Here, $\alpha_2(n)$ is defined as in condition 2 for $p = 2$.*

This linear correlation between TC_k and k, with the proportionality constant being the dimension $2n+1$, facilitates the examination of the generating function and corresponds with results in analogous spaces [8].

Our result extends the class of spaces for which the rationality conjectures holds as follows.

Theorem 4.2. *For Lens spaces L_m^{2n+1} as in Theorem 4.1 the strong rationality conjecture holds.*

Proof. Utilizing $TC_k(L_m^{2n+1}) = k \cdot (2n+1)$, we derive the generating function:

$$\begin{aligned}
\mathcal{F}_{L_m^{2n+1}}(t) &= \sum_{k=2}^{\infty} TC_k\big(L_m^{2n+1}\big) \cdot t^{k-1} \\
&= \sum_{k=2}^{\infty} k(2n+1) \cdot t^{k-1} = (2n+1) \sum_{k=2}^{\infty} k t^{k-1}.
\end{aligned}$$

We know that $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$, for $|t| < 1$. By differentiating term-by-term, we obtain:

$$\sum_{k=1}^{\infty} k t^{k-1} = \frac{1}{(1-t)^2}\,.$$

Let $r = k-1$, so:

$$\sum_{r=0}^{\infty}(r+1)t^r = \frac{1}{(1-t)^2} \qquad \text{and} \qquad \sum_{r=1}^{\infty}(r+1)t^r = \frac{1}{(1-t)^2} - 1 = \frac{t(2-t)}{(1-t)^2}\,.$$

Thus:

$$\mathcal{F}_{L_m^{2n+1}}(t) = (2n+1)\cdot\frac{t(2-t)}{(1-t)^2} = \frac{(2n+1)t(2-t)}{(1-t)^2}\,.$$

This formulation meets the requirement of Conjecture 1.2. Strong Rationality Conjecture necessitates that $P(1) = \mathrm{cat}(L_m^{2n+1})$. For the polynomial $P(t) = (2n+1)t(2-t)$ we have:

$$P(1) = (2n+1)\times 1\times(2-1) = 2n+1.$$

The Lusternik-Schnirelmann category of a Lens space corresponds to its dimension [4]:

$$\mathrm{cat}\left(L_m^{2n+1}\right) = 2n+1.$$

Given that $P(1) = \mathrm{cat}(L_m^{2n+1})$, the Strong Rationality Conjecture is confirmed. ∎

5. Conclusion and future directions

In this paper, we have treated the case in which condition (1.3) involves a unique spherical cohomology class $u \in H^r(X,\mathbb{Z})$, which is necessarily of infinite additive order. To fully extend our methodology for examining the Conjecture 1.2 for finite CW-complexes X satisfying $\mathrm{zcl}_n(X) = TC_n(X)$ for all $n \geq 2$, we need, on one hand, to address cases where $u \in H^r(X,\mathbb{Z})$ has finite order (e.g., $\mathbb{R}P^n$). On the other hand, we must consider cases involving more than one generating cohomology class in $H^r(X,\mathbb{Z})$, which may or may not be spherical. Addressing these aspects constitutes our primary focus for future research.

Furthermore, with regard to high-torsion Lens spaces L_m^{2n+1}, we demonstrate both the Weak and Strong Rationality Conjectures under the established conditions detailed in Theorem 4.1, where ([5])

$$\mathcal{F}_{L_m^{2n+1}}(x) = \frac{(2n+1)x(2-x)}{(1-x)^2} \qquad \text{and} \qquad P(1) = \mathrm{cat}\left(L_m^{2n+1}\right).$$

We intend to generalize these results to all torsion parameters and analyze Cartesian products. This work will add further clarity of the scope of rationality conjectures and their impacts in topology and robotics.

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