EXTRACTA MATHEMATICAE

Article in press
Available online March 13, 2026

On a cohomological characterization of free profinite products of three profinite groups with amalgamated subgroups

S. Douboula[1,@], N. Temate[1], G. Mantika[2], D. Tieudjo[3]

[1] *Department of Mathematics and Computer Science, Faculty of Sciences The University of Maroua, P.O. Box 814 Maroua – Cameroon*

[2] *Department of Mathematics, Higher Teacher's Training College The University of Maroua, P.O. Box 55 Maroua – Cameroon*

[3] *Department of Mathematics and Computer Science, National School of Agro-Industrial Science, The University of Ngaoundere, P.O. Box 455 Ngaoundere – Cameroon*

douboulasaliou@yahoo.fr, tematen@yahoo.fr, gilbertmantika@yahoo.fr, tieudjo@yahoo.com

Received August 12, 2025
Accepted December 31, 2025

Presented by J.M.F. Castillo

Abstract: Let $\mathcal{C}$ be the class of all finite solvable groups and $n \geq 2$ be an integer. In this paper, we presente constructions of free pro-$\mathcal{C}$ products of n pro-$\mathcal{C}$ groups with amalgamated subgroups, and of free pro-$\mathcal{C}$ products of n pro-$\mathcal{C}$ groups with commuting subgroups. We also provide conditions under which a given free pro-$\mathcal{C}$ product of three pro-$\mathcal{C}$ groups with amalgamated subgroups can be written as a free pro-$\mathcal{C}$ product with amalgamated subgroup of two free pro-$\mathcal{C}$ products with amalgamated subgroups. Furthermore, we characterize –using cohomological conditions– when a pro-$\mathcal{C}$ group is the free pro-$\mathcal{C}$ product of three of its subgroups with amalgamated subgroups. Finally, we obtain a similar characterization for free pro-$\mathcal{C}$ products of two pro-$\mathcal{C}$ groups with commuting subgroups.

Key words: pro-$\mathcal{C}$ group, pro-$\mathcal{C}$ topology, free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with amalgamation, free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups, group of continuous derivations.

MSC (2020): primary 20E06, 14G32; secondary 20E26.

1. Introduction

A *profinite group* G is the inverse limit of a projective system of finite groups, i.e., $G = \varprojlim_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a projective system of finite (abstract) groups and I is a directed set. A profinite group G is isomorphic to a closed subgroup of a direct product of finite groups. A profinite group is a topological group that is compact, Hausdorff and totally disconnected. A concrete example of a profinite group is the *profinite completion* of an abstract group. Given an abstract group G, the profinite completion $\widehat{G}$

[@] Corresponding author

ISSN: 0213-8743 (print), 2605-5686 (online)

of G is the inverse limit of the projective system $(G/N)_{N\in\mathcal{N}}$ of the (finite) quotient groups G/N, where $\mathcal{N}$ is the collection of all normal subgroups of finite index of G, i.e., $\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N$. Many authors have studied profinite groups from different perspectives [3, 5, 6, 8, 16, 18]. Luis Ribes and Pavel Zalesskii in [19] have introduced free constructions of profinite groups. They defined free profinite products of n profinite groups, amalgamated free profinite products of two profinite groups and profinite HNN-extensions of profinite groups. They also investigated the special case of proper amalgamated free profinite products and proper profinite HNN-extensions of profinite groups. They provided examples of amalgamated free profinite product which are not proper and proved some conditions for their properness [15]. Similarly, G. Mantika and D. Tieudjo defined free profinite product of profinite groups with commuting subgroups and they studied their properness (see [10]). Let G_1 and G_2 be two profinite groups, let H be a closed subgroup of G_1, K a closed subgroup of G_2 and A a closed common subgroup of G_1 and G_2. We denote by $G_1 \underset{A}{*} G_2$, $G_1 \underset{A}{\amalg} G_2$, $G_1 \underset{[H,K]}{*} G_2$ and $G_1 \underset{[H,K]}{\amalg} G_2$, the abstract amalgamation, the profinite amalgamation, the abstract free product with commuting subgroups and the free profinite product with commuting subgroups, respectively.

Today, profinite groups have been generalized to pro$-\mathcal{C}$ groups, where $\mathcal{C}$ is a class of finite groups. A pro-$\mathcal{C}$ group G is the inverse limit of a projective system of groups belonging to $\mathcal{C}$. When $\mathcal{C}$ is the class of all finite groups, all finite p-groups, all finite solvable groups and all finite nilpotent groups, then we say profinite groups, pro-p groups, pro-solvable groups and pro-nilpotent groups, respectively. Furthermore, when G is an abstract group, its profinite completion with respect to $\mathcal{C}$ is called the pro-$\mathcal{C}$ completion of group G and denoted by $\widehat{G}^{\mathcal{C}}$. Also, $\widehat{G}^{\mathcal{C}}$ is a concrete example of a pro-$\mathcal{C}$ group. Therefore, free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamation are defined (see [19]). The topology on an abstract group G given by the fundamental system of neighborhoods of the identity consisting of the collection of all its subgroups belonging in $\mathcal{C}$, is called a *pro-$\mathcal{C}$ topology* on the group G. With this topology, G becomes a topological group. A subset S of a group G *is closed* in the pro-$\mathcal{C}$ topology of G if for any element $g \in G \setminus S$, there exists a normal subgroup K of finite index in G with $G/K \in \mathcal{C}$ such that $g \notin SK$. When the trivial group is closed in the pro-$\mathcal{C}$ topology of a group G, then we say that the group G is *$\mathcal{C}$-residual*. Equivalently, G is $\mathcal{C}$-residual if for any $g \neq 1_G$ there exists a normal subgroup K in G such that $G/K \in \mathcal{C}$ and $g \notin K$. This means that, for every $g \neq 1_G$, there exists a homomorphism φ from G onto a group of $\mathcal{C}$ such that $\varphi(g) \neq 1_G$. A subgroup H of a group G is *$\mathcal{C}$-separable* if it is closed in the pro-$\mathcal{C}$ topology of G. Equivalently, a subgroup H of a group G is *$\mathcal{C}$-separable* if for any $a \in G \setminus H$, there exists a homomorphism φ from G onto a group of $\mathcal{C}$ such that $\varphi(a) \notin \varphi(H)$. D. Tieudjo in [20] recalled root-class residuality of free groups and free products of root-class residual groups. He proved some sufficient conditions for root-class residuality of generalized residual groups. Loginova in [9] proved necessary and sufficient condition such that a free product with commuting subgroups of two residually finite p-groups, is again residually finite p-group. In this paper, we study the case where $\mathcal{C}$ is the class of all finite solvable groups. We prove:

THEOREM 1.1. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1, G_2 and G_3 be three $\mathcal{C}$-residual groups, H_1 be common subgroup of G_1 and G_2 and let H_2 be common subgroups of G_2 and G_3 such that G_2 is generated by H_1 and H_2. Assume that H_1 is central in G_1, H_2 is abelian and commute with G_1. Then, $G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$, the free product of G_1, G_2 and G_3 with amalgamated subgroups H_1 and H_2, is $\mathcal{C}$-residual if and only if the subgroups H_1 and H_2 are $\mathcal{C}$-separable in G_1 and G_3 respectively.*

Also, when studying the residual finiteness of free products of abstract groups with commuting subgroups, Loginova in [9] established that this construction can be written as double amalgamation. That is, given G_1 and G_2 two abstract groups with respective subgroups H and K, the following situation holds:

$$G_1 \underset{[H,K]}{*} G_2 = \Big(G_1 \underset{H}{*} (H \times K)\Big) \underset{H\times K}{*} \Big((H \times K) \underset{K}{*} G_2\Big).$$

For profinite groups or pro$-\mathcal{C}$ groups in general, this is not always true. Let $\mathcal{C}$ be the class of all finite solvable groups. Then, $\mathcal{C}$ is subgroup closed and is also closed under taking quotients, under forming finite direct products, under extensions, and for any group G with normal subgroups H and K such that $G/H, G/K \in \mathcal{C}$, then $G/H \cap K \in \mathcal{C}$. See [19, 20]. The class $\mathcal{C}$ is a root-class (see [20]). In this paper, under some conditions, we write the free pro-$\mathcal{C}$ product with commuting subgroups of pro-$\mathcal{C}$ groups as a pro$-\mathcal{C}$ product with amalgamation of two pro$-\mathcal{C}$ products with amalgamation. That is:

THEOREM 1.2. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1, G_2 and G_3 be three pro$-\mathcal{C}$ groups, H_1 be common subgroup of G_1 and G_2*

and let H_2 be common subgroups of G_2 and G_3 such that G_2 is generated by H_1 and H_2. Assume that the pro–$\mathcal{C}$ topology of $G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ induces on G_1, G_2, G_3, H_1 and on H_2 their pro–$\mathcal{C}$ topologies. If H_1 is central in G_1, H_2 is abelian and commute with G_1, and H_1 and H_2 are $\mathcal{C}$-separable and satisfy $\widehat{G_2}^{\mathcal{C}} = G_2$, then we have:

$$G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3 = \Big(G_1 \underset{H}{\amalg} G_2\Big) \underset{G_2}{\amalg} \Big(G_2 \underset{K}{\amalg} G_3\Big).$$

Some free constructions of groups (abstract or topological case) were also characterized with cohomology tools [14, 16, 17]. Let $\mathcal{C}$ be the class of all finite solvable groups. The pro-$\mathcal{C}$ completion of $\mathbb{Z}$, the ring of integers, is the free pro-$\mathcal{C}$ group on a single generator noted by $\mathbb{Z}_{\hat{\mathcal{C}}}$. It has an obvious structure of a compact, Haussdorff ring (see [7]). Let R be a commutative ring and let G be a pro-$\mathcal{C}$ group. The abstract group algebra (or group ring) is denoted by $[RG]$. The complete group algebra of G is defined by $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]] = \varprojlim_U [\mathbb{Z}_{\hat{\mathcal{C}}}G/U]$, where U runs through the open normal subgroups of G. $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]$ is a profinite ring. Throughout this paper, $\mathrm{DMod}([[\mathbb{Z}_{\hat{\mathcal{C}}}G]])$ denotes the category of discrete $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]$-modules.

Let now M be a closed subgroup of G. For $A \in \mathrm{DMod}([[\mathbb{Z}_{\hat{\mathcal{C}}}G]])$, define

$$\mathrm{Der}_M(G, A) = \big\{d : G \to A : d(xy) = xd(y) + d(x),\, \forall\, x, y \in G,\, d|_M = 0\big\},$$

the group of all continuous derivations from G to A vanishing on M. L. Ribes and P. Zalesskii characterized cohomologically free pro-$\mathcal{C}$ products of two pro-$\mathcal{C}$ groups with amalgamation, where $\mathcal{C}$ is an extension closed variety of finite solvable groups. See [19, Theorem 9.3.1]. In this paper, following L. Ribes and P. Zalesskii, we obtain an analogous criterion in terms of derivations, when a pro-$\mathcal{C}$ group G is a free pro-$\mathcal{C}$ product with amalgamated subgroups of three of its subgroups. We prove:

Theorem 1.3. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G be a pro-$\mathcal{C}$ group. Let G_1, G_2, G_3, H_1 and H_2 be closed subgroups of G such that G_2 is generated by H_1 and H_2. Assume that the pro–$\mathcal{C}$ topology of G induces on G_1, G_2, G_3, H_1 and on H_2 their pro–$\mathcal{C}$ topologies. If H_1 is a common subgroup of G_1 and G_2, H_2 is a common subgroup of G_2 and G_3 such that H_1 is central in G_1, H_2 is abelian and commute with G_1, and H_1 and H_2 are $\mathcal{C}$-separable and $\widehat{G_2}^{\mathcal{C}} = G_2$, then the following conditions are equivalent:*

(i) $G = G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3$;

(ii) *The natural homomorphism*

$$\begin{array}{rcl}\psi_G : \mathrm{Der}_{G_2}(G, A) & \longrightarrow & \mathrm{Der}_{G_2}(G_1 \amalg H_2, A) \times \mathrm{Der}_{G_2}(G_3 \amalg H_1, A) \\ f & \longmapsto & \big(f|_{G_1 \amalg H_2}, f|_{G_3 \amalg H_1}\big),\end{array}$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]-$*modules* $A \in \mathcal{C}$.

Corollary 1.4. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G be a pro-$\mathcal{C}$ group. Let G_1 and G_2 be closed subgroups of G. If H and K are respective subgroups of G_1 and G_2 such that H is central in G_1, K is abelian and commute with G_1, and H and K are $\mathcal{C}$-separable and satisfy $\widehat{H \times K}^{\mathcal{C}} = H \times K$, then the following conditions are equivalent:*

1. $G = G_1 \underset{[H,K]}{\amalg} G_2$ *(free pro-$\mathcal{C}$ product with commuting subgroups);*
2. *The natural homomorphism*

$$\begin{array}{rcl}\psi_G : \mathrm{Der}_{H \times K}(G, A) & \longrightarrow & \mathrm{Der}_{H \times K}(G_1 \amalg K, A) \times \mathrm{Der}_{H \times K}(G_2 \amalg H, A) \\ f & \longmapsto & \big(f|_{G_1 \amalg K}, f|_{G_2 \amalg H}\big),\end{array}$$

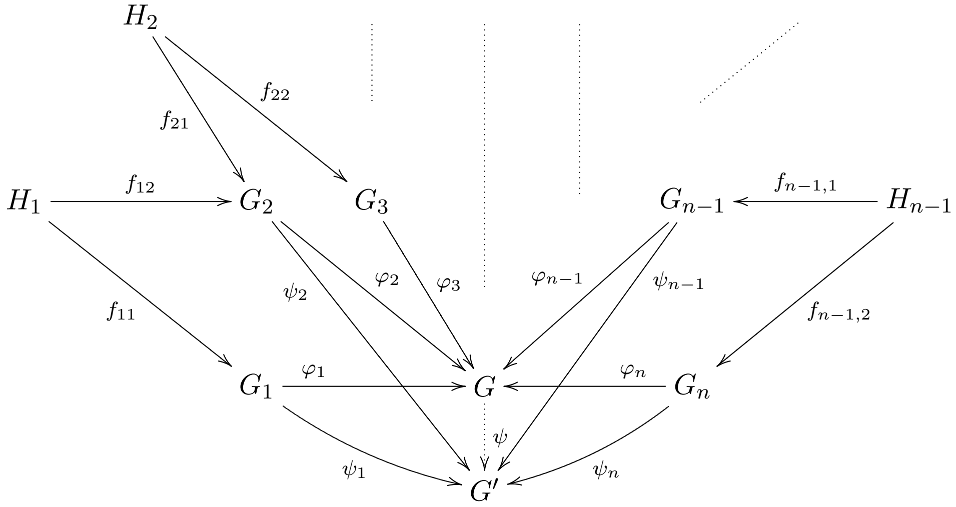
is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]-$*modules* $A \in \mathcal{C}$.

2. Preliminaries notions and results

In this section, we give definitions and properties of some notions we will use. One can refer to [2, 10, 19, 4] for more details on the notions of *profinite topology*, *retract semidirect factor*, *compatibility* or *filtration*.

2.1. Some free construction of pro-$\mathcal{C}$ groups

Definition 2.1. (Pushout: Free pro$-\mathcal{C}$ product of n pro-$\mathcal{C}$ groups with amalgamation.) Let $n \geq 2$ be integer, G_1, G_2, $\ldots$, G_n be pro$-\mathcal{C}$ groups and H_1, H_2, $\ldots$, H_{n-1} such that for every $i \in \{1, \ldots, n-1\}$, H_i is a common closed subgroup of G_i and G_{i+1}. Let $f_{i1} : H_i \to G_i$ and $f_{i2} : H_i \to G_{i+1}$ be the inclusion maps. The free pro$-\mathcal{C}$ product of pro$-\mathcal{C}$ groups G_1, G_2, $\ldots$, G_n with amalgamated subgroups H_1, H_2, $\ldots$, H_{n-1} is defined to be a pushout in the category of pro$-\mathcal{C}$ groups, i.e. a pro$-\mathcal{C}$ group G together with continuous homomorphisms $\varphi_i : G_i \to G$ $(i = 1, 2, \ldots, n)$, satisfying the following universal property: for any continuous homomorphisms $\psi_i : G_i \to G'$ into a pro$-\mathcal{C}$ group G' with $\psi_i f_{i1} = \psi_{i+1} f_{i2}$, there exists a unique continuous homomorphism $\psi : G \to G'$ such that $\psi\varphi_i = \psi_i$ i.e. the following diagram is commutative:



Remark 2.2. Since a pro$-\mathcal{C}$ group is a projective limit of a projective system of groups in $\mathcal{C}$, it is enough to consider G' in $\mathcal{C}$ to check the universal property in the previous definition.

A concrete free pro$-\mathcal{C}$ product of n pro$-\mathcal{C}$ groups $G_1, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_{n-1}$ can be constructed as follows:

Let $n \geq 2$ be integer, $G_1, G_2, \ldots, G_n$ be pro$-\mathcal{C}$ groups and $H_1, H_2, \ldots, H_{n-1}$ such that for every $i \in \{1, \ldots, n-1\}$, H_i is a common closed subgroup of G_i and G_{i+1}. Let $f_{i1} : H_i \to G_i$ and $f_{i2} : H_i \to G_{i+1}$ be continuous monomorphisms. Then one can construct the abstract free product $\tilde{G}$ of abstract groups $G_1, G_2, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_{n-1}$ i.e. $\tilde{G} = G_1 \underset{H_1}{*} \cdots \underset{H_{n-1}}{*} G_n$; see [13] for more details. We have the inclusions $\tilde{\varphi} : G_i \to \tilde{G}$, for every $i = 1, \ldots, n$. Now any G_i can be identified to its image in the group $\tilde{G}$.

Let $\mathcal{N} = \{N \lhd_f \tilde{G} : N \cap G_i \text{ is open in } G_i,\ i = 1, \ldots, n \text{ and } \tilde{G}/N \in \mathcal{C}\}$. Let now $\widehat{G} = \varprojlim_{N \in \mathcal{N}} \tilde{G}/N$ be the pro$-\mathcal{C}$ completion of the abstract group $\tilde{G}$. Let $\tilde{\varphi} : \tilde{G} \to \widehat{G}$ be the canonical homomorphism. Then for any $i = 1, \ldots, n$ we have $\varphi_i = \tilde{\varphi}\tilde{\varphi}_i : G_i \to \widehat{G}$ is a homomorphism. So, the family $(\widehat{G}, \varphi_i)_{i=1,\ldots,n}$ is the free pro$-\mathcal{C}$ product of the pro$-\mathcal{C}$ groups $G_1, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_{n-1}$.

Indeed, $\tilde{\varphi}_i f_{i1}(H_i) = \tilde{\varphi}_{i+1} f_{i2}(H_{i+1})$ from the construction of $\tilde{G}$ and since $\tilde{\varphi}$ is a group homomorphism then $\tilde{\varphi}\tilde{\varphi}_i f_{i1}(H_i) = \tilde{\varphi}\tilde{\varphi}_{i+1} f_{i2}(H_{i+1})$. Thus, $\varphi_i f_{i1}(H_i) = \varphi_{i+1} f_{i2}(H_{i+1})$.

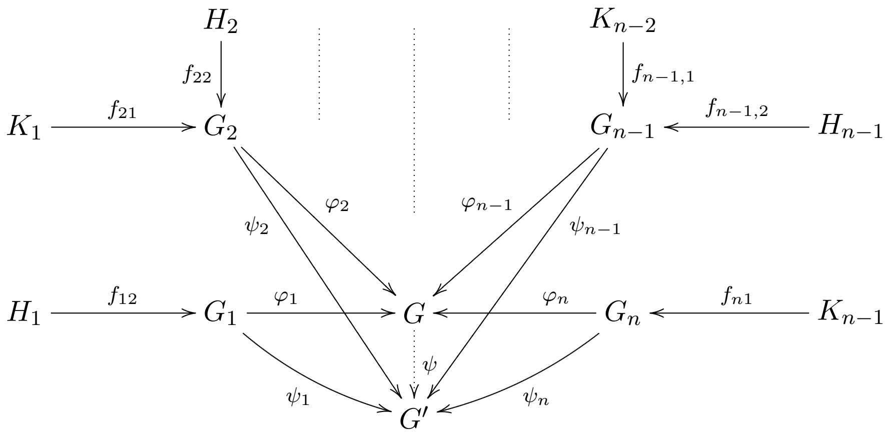
Let now G' be a group in $\mathcal{C}$, $\psi_i : G_i \to G'$ be continuous homomorphism such that $\psi_i f_{i1}(H_i) = \psi_{i+1} f_{i2}(H_{i+1})$. By the universal property of $\tilde{G}$, there exists a unique group homomorphism $\tilde{\psi} : \tilde{G} \to G'$ satisfying $\psi_i = \tilde{\psi}\tilde{\varphi}_i$, $i = 1, \ldots, n$. We have $(\tilde{\varphi}_i)^{-1}(\operatorname{Ker}\tilde{\psi}) = \operatorname{Ker}\psi_i$, $i = 1, \ldots, n$. Since G' is Hausdorff, then $\{1_{G'}\}$ is closed. Moreover G' is compact; thus $\{1_{G'}\}$, as closed subgroup of finite index, is open. So, for $i = 1, \ldots, n$, $\operatorname{Ker}\psi_i = (\psi_i)^{-1}(\{1_{G'}\})$ is open i.e. $(\tilde{\varphi}_i)^{-1}(\operatorname{Ker}\tilde{\psi})$ is open in G_i. Thus $\operatorname{Ker}\tilde{\psi} \in \mathcal{N}$. Let U be an open normal subgroup of finite index of G'. Then U is an open neighbourhood of $\{1_{G'}\}$, and we trivially have that the image of $\operatorname{Ker}\tilde{\psi}$ by $\tilde{\psi}$ is contained in U. So $\tilde{\psi}$ is continuous, since it is continuous on $\{1_{G'}\}$. Then, by the definition of $\widehat{G}$, there is a continuous homomorphism $\psi : \widehat{G} \to G'$ satisfying $\tilde{\psi} = \psi\tilde{\varphi}$. Hence, we have $\psi\varphi_i = \psi\tilde{\varphi}\tilde{\varphi}_i = \psi_i$ for $i = 1, \ldots, n$. Since $\widehat{G}$ is the pro$-\mathcal{C}$ completion of the abstract group $\tilde{G}$ which is generated by groups $G_1, \ldots, G_n$, then $\widehat{G} = \overline{< \varphi_1(G_1), \ldots, \varphi_n(G_n) >}$. Consequently, ψ is unique. Now , $(\widehat{G}, \varphi_i)_{i=1,\ldots,n}$ is the free pro$-\mathcal{C}$ product of the pro$-\mathcal{C}$ groups $G_1, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_{n-1}$.

PROPOSITION 2.3. *The free pro$-\mathcal{C}$ product of n pro$-\mathcal{C}$ groups $G_1, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_n$ is unique up to isomorphism.*

So, $G = G_1 \coprod_{H_1} \cdots \coprod_{H_{n-1}} G_n$ will be denote the free pro$-\mathcal{C}$ product of pro$-\mathcal{C}$ groups $G_1, \ldots, G_n$ with amalgamated subgroups $H_1, \ldots, H_n$. Note that the free abstract product of n abstract groups is defined in the same way as the previous definition, just omitting the continuity of the morphisms and is denoted by $G = G_1 \ast_{H_1} \cdots \ast_{H_{n-1}} G_n$.

DEFINITION 2.4. (Free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups.) Let $n \geq 2$ be integer, G_1, G_2, $\ldots$, G_n be pro-$\mathcal{C}$ groups, H_1 be closed subgroup of G_1, K_{n-1} be closed subgroup of G_n and for $i = 1, \ldots, n-2$, K_i, H_{i+1} be closed subgroups of G_{i+1}. Let $f_{i1} : K_{i-1} \to G_i$, $f_{i2} : H_i \to G_i$ for every $i = 1, \ldots, n$ be inclusion maps (note that f_{i1} and f_{i2} do not exist when $i = 1$ and $i = n$ respectively). A free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups G_1, G_2, $\ldots$, G_n with commuting subgroups H_1 and K_1, H_2 and K_2, $\ldots$, H_{n-1} and K_{n-1} is a pro-$\mathcal{C}$ group G together with continuous homomorphisms $\varphi_i : G_i \to G$ such that $[\varphi_i(f_{i2}(H_i)); \varphi_{i+1}(f_{i+1,1}(K_i))] = 1$ for $i = 1, \ldots, n$, satisfying the following universal property: for any continuous homomorphisms $\psi_i : G_i \to G'$

into a pro-$\mathcal{C}$ group G' with $\big[\psi_i(f_{i2}(H_i)); \psi_{i+1}(f_{i+1,1}(K_i))\big] = 1$ there exists a unique continuous homomorphism $\psi : G \to G'$ such that $\psi\varphi_i = \psi_i$. This situation can be illustrated by the following commutative diagram:



The free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups G_1, G_2, ..., G_n with commuting subgroups H_1 and K_1, H_2 and K_2, ..., H_{n-1} and K_{n-1} is unique (up to isomorphism) and we denote this group by $G_1 \underset{[H_1,K_1]}{\amalg} G_2 \underset{[H_2,K_2]}{\amalg} \cdots \underset{[H_{n-1},K_{n-1}]}{\amalg} G_n$. Note that the free abstract product of n abstract groups with commuting subgroups is defined in the same way as the previous definition, just omitting the continuity of the morphisms and is denoted by

$$G_1 \underset{[H_1,K_1]}{*} G_2 \underset{[H_2,K_2]}{*} \cdots \underset{[H_{n-1},K_{n-1}]}{*} G_n.$$

The construction of free pro-$\mathcal{C}$ product of n pro-$\mathcal{C}$ groups with commuting subgroups is similar as the construction of the free pro-$\mathcal{C}$ product of n pro-$\mathcal{C}$ groups with amalgamated subgroups presented above, i.e.,

$$G_1 \underset{[H_1,K_1]}{\amalg} \cdots \underset{[H_{n-1},K_{n-1}]}{\amalg} G_n = \overbrace{G_1 \underset{[H_1,K_1]}{*} \cdots \underset{[H_{n-1},K_{n-1}]}{*} G_n}.$$

Proposition 2.5. *Let G_1 and G_2 be two pro–$\mathcal{C}$ groups with respective closed subgroups H and K. Then, the pro-$\mathcal{C}$ topology of $G = G_1 \underset{[H,K]}{*} G_2$ induces on G_1, G_2, H and K their pro-$\mathcal{C}$ topologies.*

Proof. To prove that the pro-$\mathcal{C}$ topology of G induces on G_2 (for example, and the similar reason for G_1, H and K) its pro-$\mathcal{C}$ topology, it suffices to prove that G_2 is the retract semidirect factor of G. Indeed, assume that G_2 is a retract semidirect factor of G and let prove that the pro-$\mathcal{C}$ topology of G induces on G_2 its pro-$\mathcal{C}$ topology. So, let M_2 be a normal subgroup of G_2 of finite index such that $G_2/M_2 \in \mathcal{C}$. Since G_2 is a retract semidirect factor of G, there exists A, a normal subgroup of G such that $G = A \rtimes G_2$. Clearly $AM_2 \lhd_f G$ since

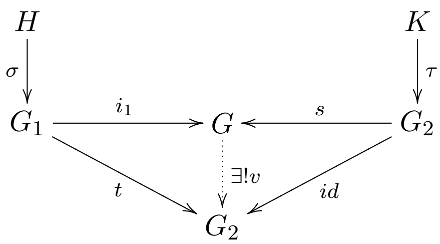
$$G_2/M_2 = G_2/(AM_2) \cap G_2 \simeq (AM_2)G_2/AM_2 = G/AM_2,$$

where $(AM_2) \cap G_2 = M_2$.

Therefore $G/AM_2 \in \mathcal{C}$, and it follows then that the pro-$\mathcal{C}$ topology of G induces on G_2 its pro-$\mathcal{C}$ topology.

Let now prove that G_2 is a retract semidirect factor of G. To do it, we will build a homomorphism $v : G \to G_2$ with $v \circ s = id_{G_2}$, where $s : G_2 \to G$ is the canonical homomorphism.

Since $G = G_1 \underset{[H,K]}{*} G_2$, so by the definition of free products, there is a (canonical) map $s : G_2 \to G$, including G_2 as a subgroup. By the universal property of free products, there exists a unique homomorphism $v : G \to G_2$ defined by the identity map $id : G_2 \to G_2$ and the trivial map $t : G_1 \to G_2$. Clearly, K commutes with the identity element, which is the image of H. This situation is illustrated by the following commutative diagram.



Therefore, G_2 is a retract semidirect factor of G and the proposition is proven. ∎

2.2. Cohomology with coefficients in discrete modules Let G be a pro-$\mathcal{C}$ group. In the context of pro$-\mathcal{C}$ groups, the analogue of the group ring is the concept of complete group algebra.

Definition 2.6. Let G be a pro-$\mathcal{C}$ group and R a profinite ring. The complete group algebra of G denoted by $[[RG]]$ is defined by

$$[[RG]] = \varprojlim_{U} [RG/U],$$

where U runs through the open normal subgroups of G.

Then, $[[RG]]$ is a profinite ring since we can express it as an inverse limit of finite rings, i.e.,

$$[[RG]] = \varprojlim [(R/I)(G/U)],$$

where I and U range over the open ideals of R and the open normal subgroups of G, respectively, see [19]. Recall also that every $[[RG]]$-module is a G-module (see [19, Proposition 5.3.6]). DMod($[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]$) denotes the category of discrete $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]$-modules. Let G be a pro-$\mathcal{C}$ group and let A be a discrete G-module. Let $C^n(G,A)$ be the (abelian) group of all continuous functions $f : G^n \to A$. Define a cochain complex

$$0 \to C^0(G,A) \longrightarrow C^1(G,A) \longrightarrow \cdots \longrightarrow C^n(G,A) \stackrel{\partial^{n+1}}{\longrightarrow} C^{n+1}(G,A) \longrightarrow \dots,$$

where ∂^{n+1} is defined as follows:

$$\begin{aligned}(\partial^{n+1} f)(x_1,\dots,x_{n+1}) = x_1 f(x_2,\dots,x_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(x_1,\dots,x_i x_{i+1},\dots,x_{n+1}) \\ + (-1)^{i+1} f(x_1,\dots,x_n),\end{aligned}$$

with $x_1,\dots,x_{n+1} \in G$.

Definition 2.7. Let G be a pro-$\mathcal{C}$ group and let A be a discrete G-module. Then the $n-$th cohomology group of G with coefficients in A is defined as the $n-$th cohomology group of the above cochain complex, i.e.,

$$H^n(G,A) = \frac{\operatorname{Ker}(\partial^{n+1})}{\operatorname{Im}(\partial^n)}.$$

According to above definition, $H^1(G,A) = \operatorname{Ker}(\partial^2)/\operatorname{Im}(\partial^1)$. The elements of $\operatorname{Ker}(\partial^2)$ are called *crossed homomorphisms* or *derivations* from G to A. So, a derivation $d : G \to A$ is a continuous function such that $d(xy) = xd(y)+d(x)$, for all $x, y \in G$. We denote by $\operatorname{Der}(G,A)$, the (abelian) group of derivations. The elements of $\operatorname{Im}(\partial^1)$ are called *principal crossed homomorphisms* or *inner derivations.*

3. Proof of Theorem 1.1

The following property is the extension to $\mathcal{C}$-groups of [1, Theorem 2].

Lemma 3.1. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let $G = A \underset{H}{*} B$ be the free product of A and B, two groups of $\mathcal{C}$, with amalgamated subgroup H. If H is central in A or in B, then G is $\mathcal{C}$-residual.*

Proof. Let A and B be groups of $\mathcal{C}$ with a common subgroup H. Suppose that H is central in A or in B. Let $G = A \underset{H}{*} B$ be the free product of A and B with amalgamated subgroup H. Using simultaneously [11, Corollary 15.2, p. 532] and [12, Theorem 4., p. 11], there is a finite group G_1 of $\mathcal{C}$ containing isomorphic copies A_1 and B_1 of A and B, respectively, with isomorphisms

$$\theta : A \longrightarrow A_1; \qquad \phi : B \longrightarrow B_1.$$

G_1 can be chosen such that the isomorphisms θ and ϕ coincide on H, see [11, p. 532].

Since G is the free product of A and B with amalgamated subgroup H, it follows that θ and ϕ can be simultaneously extended to a homomorphism μ of G onto G_1. Let $K = \mathrm{Ker}(\mu)$. Since G_1 is finite, it follows that K is of finite index in G. In accordance to [1, Theorem 2], K is free. Consequently, K is $\mathcal{C}$-residual ([20], Theorem 2.1) since $\mathcal{C}$ is a root-class. It follows that G is $\mathcal{C}$-residual as a finite extension of a $\mathcal{C}$-residual group, and the Lemma is demonstrated. ∎

Proof of Theorem 1.1. Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1, G_2 and G_3 be three $\mathcal{C}$-residual groups, H_1 be common subgroup of G_1 and G_2 and let H_2 be common subgroups of G_2 and G_3 such that G_2 is generated by H_1 and H_2. Assume that H_1 is central in G_1, H_2 is abelian and commute with G_1. We first write G as a double amalgamation. That is,

$$G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3 = \Big(G_1 \underset{H_1}{*} G_2\Big) \underset{G_2}{*} \Big(G_2 \underset{H_2}{*} G_3\Big).$$

1. Assume that the subgroups H_1 and H_2 are not $\mathcal{C}$-separable, and let us prove that G is not $\mathcal{C}$-residual. Because H_2 is not $\mathcal{C}$-separable in G_3, there exists an element $a \in G_3 \setminus H_2$ and a homomorphism η onto a finite group in $\mathcal{C}$ such that $\eta(a) \in \eta(H_2)$. Let h be a non-identity element of H_2. Then the

element $w = [a, h]$ of the group $G_2 \underset{H_2}{*} G_3$ differs from 1, since $w = a^{-1}h^{-1}ah$ is reduced in the free product with amalgamation $G_2 \underset{H_2}{*} G_3$. Clearly, the image of this element under any homomorphism of $G_2 \underset{H_2}{*} G_3$ onto a finite group in $\mathcal{C}$ equals 1. Thus, $G_2 \underset{H_2}{*} G_3$ is not $\mathcal{C}$-residual and likewise G.

2. Conversely, let the subgroups H_1 and H_2 be $\mathcal{C}$-separable in G_1 and G_3, respectively. Let us prove that the group $G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ is $\mathcal{C}$-residual. Consider $(R_i)_{i\in I}$, the family of all normal subgroups of finite index in $G_1 \underset{H_1}{*} G_2$ with $G_1 \underset{H_1}{*} G_2/R_i \in \mathcal{C}$ for all $i \in I$ and let $(S_j)_{j\in J}$ be the family of all normal subgroups of finite index in $G_2 \underset{H_2}{*} G_3$ with $G_2 \underset{H_2}{*} G_3/R_j \in \mathcal{C}$ for all $j \in J$. Denote by Λ the subset of $I \times J$ that consists of the various pairs (i, j) such that the subgroups R_i and R_j are (G_2, G_2)-compatible and put $R_\lambda = R_i$ and $S_\lambda = S_j$ for an arbitrary element $\lambda = (i, j) \in \Lambda$. Since the groups G_1 and G_3 are $\mathcal{C}$-residual and their subgroups H_1 and H_2 are $\mathcal{C}$-separable, it follows from [9, Lemma 1] that for every non-identity element g of $G_1 \underset{H_1}{*} G_2$, there exists an element $\lambda_g \in \Lambda$ such that $g \notin R_{\lambda_g}$. Moreover, if g does not belong to G_2, the subgroup R_{λ_g} can be chosen so that $g \notin G_2R_{\lambda_g}$. Consequently, $\underset{\lambda\in\Lambda}{\cap} R_\lambda = 1$ and $\underset{\lambda\in\Lambda}{\cap} G_2R_\lambda = G_2$. Therefore, the family $(R_\lambda)_{\lambda\in\Lambda}$ is a G_2-filtration. Similarly, we obtain that the family $(S_\lambda)_{\lambda\in\Lambda}$ is a G_2-filtration. Thus, for every $\lambda \in \Lambda$, the map

$$G_2R_\lambda/R_\lambda \longrightarrow G_2S_\lambda/S_\lambda$$

from the subgroup G_2R_λ/R_λ of the quotient group $G_1 \underset{H_1}{*} G_2/R_\lambda$ onto the subgroup G_2S_λ/S_λ of the quotient group $G_2 \underset{H_2}{*} G_3/S_\lambda$, determined by the rule $\varphi_{R_\lambda,S_\lambda}(xR_\lambda) = xS_\lambda$ $(x \in G_2)$, is well defined and clearly an isomorphism since R_λ and S_λ are (G_2, G_2)-compatible. Therefore, we construct the group

$$G_{R_\lambda,S_\lambda} = G_1 \underset{H_1}{*} G_2/R_\lambda \underset{G_2R_\lambda/R_\lambda = G_2S_\lambda/S_\lambda}{*} G_2 \underset{H_2}{*} G_3/S_\lambda.$$

The natural mappings from the group $G_1 \underset{H_1}{*} G_2$ onto the quotient group $G_1 \underset{H_1}{*} G_2/R_\lambda$ and from the group $G_2 \underset{H_2}{*} G_3$ onto the quotient group $G_2 \underset{H_2}{*} G_3/S_\lambda$ extend to a homomorphism $\rho_{R_\lambda,S_\lambda}$ from the group

$$G = \left(G_1 \underset{H_1}{*} G_2\right) \underset{G_2}{*} \left(G_2 \underset{H_2}{*} G_3\right)$$

onto the group $G_\lambda = G_{R_\lambda,S_\lambda}$.

Note that the families $(R_\lambda)_{\lambda\in\Lambda}$ and $(S_\lambda)_{\lambda\in\Lambda}$ are closed under finite intersections, i.e., for any λ_1, $\lambda_2 \in \Lambda$, there is an index $\lambda \in \Lambda$ such that $R_{\lambda_1} \cap R_{\lambda_2} = R_\lambda$ and $S_{\lambda_1} \cap S_{\lambda_2} = S_\lambda$.

Therefore, if g is a nonidentity element of G, then, considering a reduced form of g and the fact that the families $(R_\lambda)_{\lambda\in\Lambda}$ and $(S_\lambda)_{\lambda\in\Lambda}$ forms a G_2-filtration, there exists $\lambda \in \Lambda$ such that the image of g under the homomorphism $\rho_\lambda = \rho_{R_\lambda,S_\lambda}$ is not equal to 1. Indeed, let $g \in G$.

• If $g \in G_1 \underset{H_1}{*} G_2$, put $\lambda \in \Lambda$ such that $g \notin R_\lambda$. Then, $\rho_\lambda(g) = gR_\lambda \neq R_\lambda$. Similarly, we prove that there exists $\lambda \in \Lambda$ such that $\rho_\lambda(g) = gS_\lambda \neq S_\lambda$ if $g \in G_2 \underset{H_2}{*} G_2$.

• If $g \notin \left(G_1 \underset{H_1}{*} G_2\right) \cup \left(G_2 \underset{H_2}{*} G_3\right)$, write $g = x_1y_1x_2y_2 \ldots x_ny_n$ with $x_i \in G_1 \underset{H_1}{*} G_2$, $y_i \in G_2 \underset{H_2}{*} G_3$, $x_i, y_i \notin G_2$, $1 \leq i \leq n$. We choose a suitable $\lambda \in \Lambda$ such that $x_i \notin G_2R_\lambda$ and $y_i \notin G_2S_\lambda$, $1 \leq i \leq n$ as follows.

Put $\lambda_1 \in \Lambda$ such that $x_i \notin G_2R_{\lambda_1}$. Note that this choice is possible since $(R_\lambda)_{\lambda\in\Lambda}$ is G_2-filtration and closed under finite intersection. Similarly, put $\lambda_2 \in \Lambda$ such that $y_i \notin G_2S_{\lambda_2}$, $1 \leq i \leq n$. Then take $\lambda \in \Lambda$ such that $R_\lambda = R_{\lambda_1} \cap R_{\lambda_2}$, $S_\lambda = S_{\lambda_1} \cap S_{\lambda_2}$.

See that, $\rho_\lambda(g) = x_1R_\lambda y_1S_\lambda x_2R_\lambda y_2S_\lambda \cdots x_nR_\lambda y_nS_\lambda \neq R_\lambda = S_\lambda$ in G_λ.

By Lemma 3.1, $G_\lambda = G_{R_\lambda,S_\lambda}$ is $\mathcal{C}$-residual. Indeed, G_λ is the free product of two groups of $\mathcal{C}$ with amalgamated subgroup $G_2R_\lambda/R_\lambda = G_2S_\lambda/S_\lambda$ which is central in $G_1 \underset{H_1}{*} G_2/R_\lambda$.

To see that $G_2R_\lambda/R_\lambda = G_2S_\lambda/S_\lambda$ is central in $G_1 \underset{H_1}{*} G_2/R_\lambda$, let $u \in G_1 \underset{H_1}{*} G_2/R_\lambda$, so that $u = xR_\lambda$ with $x \in G_1 \underset{H_1}{*} G_2$. Assume that $x = g_1k_1g_2k_2 \cdots g_nk_n$, in its reduced form in $G_1 \underset{H_1}{*} G_2/R_\lambda$ with $g_i \in G_1$ and $k_i \in H_2$. Let $v = yR_\lambda \in G_2R_\lambda/R_\lambda$ with $y = hk \in G_2$ ($h \in H_1$, $k \in H_2$). Then,

$$\begin{aligned} uv &= xR_\lambda yR_\lambda = xyR_\lambda = g_1k_1g_2k_2 \cdots g_nk_nhkR_\lambda \\ &= hkg_1k_1g_2k_2 \cdots g_nk_nR_\lambda \text{ (since } H_1 \text{ is central in } G_1, H_2 \\ &\qquad \text{is abelian and commute with } G_1) \\ &= yxR_\lambda = yR_\lambda xR_\lambda = vu. \end{aligned}$$

Now, since $\rho_\lambda(g) \neq 1$ and G_λ is $\mathcal{C}$-residual, it follows that there exists a homomorphism l from G_λ to a group of $\mathcal{C}$ such that for every nonidentity element g of G we have $l\rho_\lambda(g) \neq 1$, a nonidentity image. Consequently, G is $\mathcal{C}$-residual. ∎

4. Proof of Theorem 1.2

We first prove the following lemma.

Lemma 4.1. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1, G_2 and G_3 be three pro–$\mathcal{C}$ groups, let H_1 be a common subgroup of G_1 and G_2 and let H_2 be a common subgroups of G_2 and G_3. Assume that the pro–$\mathcal{C}$ topology of $G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ induces on G_1, G_2, G_3, H_1 and on H_2 their respective pro–$\mathcal{C}$ topologies. Then, the pro-$\mathcal{C}$ topology of G induces on $G_1 \underset{H_1}{*} G_2$ and $G_2 \underset{H_2}{*} G_3$ their respective pro-$\mathcal{C}$ topologies.*

Proof. Set

$$\mathcal{N} = \{N \lhd_f G : N \cap G_i \text{ is open in } G_i,\ i = 1, 2, 3 \text{ and } G/N \in \mathcal{C}\}$$

$$\mathcal{N}_1 = \left\{N \lhd_f G_1 \underset{H_1}{*} G_2 : N \cap G_i \,\text{is open in}\, G_i,\ i = 1, 2,\ \&\ \left(G_1 \underset{H_1}{*} G_2\right)/N \in \mathcal{C}\right\},$$

$$\mathcal{N}_2 = \left\{N \lhd_f G_2 \underset{H_2}{*} G_3 : N \cap G_i \,\text{is open in}\, G_i,\ i = 2, 3,\ \&\ \left(G_2 \underset{H_2}{*} G_3\right)/N \in \mathcal{C}\right\},$$

$$\mathcal{N}_{induced} = \left\{N \cap \left(G_1 \underset{H_1}{*} G_2\right)\ :\ N \in \mathcal{N}\right\}.$$

Let us prove that $\mathcal{N}_1 = \mathcal{N}_{induced}$.

(1) Clearly, $\mathcal{N}_{induced} \subset \mathcal{N}_1$.

(2) We now prove that $\mathcal{N}_1 \subset \mathcal{N}_{induced}$. Let $N \in \mathcal{N}_1$. We want to find $M \in \mathcal{N}$ such that $M \cap \left(G_1 \underset{H_1}{*} G_2\right) = N$. It is enough to find $M' \in \mathcal{N}$ such that $M' \cap \left(G_1 \underset{H_1}{*} G_2\right) \leq N$. Indeed, if such $M' \in \mathcal{N}$ exists, then $M' \cap \left(G_1 \underset{H_1}{*} G_2\right) \in \mathcal{N}_{induced}$, and consequently, $N \in \mathcal{N}_{induced}$ as a subgroup of $G_1 \underset{H_1}{*} G_2$ containing the non-empty open set $M' \cap \left(G_1 \underset{H_1}{*} G_2\right)$. It follows then that there exists $M \in \mathcal{N}$ such that $M \cap \left(G_1 \underset{H}{*} (H \times K)\right) = N$ as required.

We now construct $M' \in \mathcal{N}$ such that $M' \cap \left(G_1 \underset{H_1}{*} G_2\right) \leq N$.

Clearly, $N \cap G_1$ is open in G_1, $N \cap G_1 \lhd_f G_1$ and $G_1/N \cap G_1 \in \mathcal{C}$. Since the pro-$\mathcal{C}$ topology of G induces on G_1 its pro-$\mathcal{C}$ topology, it follows that there exists $M_1 \in \mathcal{N}$ such that $M_1 \cap G_1 = N \cap G_1$. Similarly, there exists $M_2 \in \mathcal{N}$ such that $M_2 \cap G_2 = N \cap G_2$. Set $M' = M_1 \cap M_2$.

(a) We now show that $M' \in \mathcal{N}$.

(i) It is obvious that $M' \lhd_f G$

(ii) For any $i = 1, 2, 3$, we have

$$M' \cap G_i = M_1 \cap M_2 \cap G_i = (M_1 \cap G_i) \cap (M_2 \cap G_i).$$

Since M_1 and M_2 belong to $\mathcal{N}$, the subgroups $M_1 \cap G_i$ and $M_2 \cap G_i$ are open in G_i and so is their intersection $M' \cap G_i$.

(iii) We now prove that $G/M' \in \mathcal{C}$.
Since $G/M_1 \in \mathcal{C}$ and $G/M_2 \in \mathcal{C}$, then $G/M_1 \cap M_2 \in \mathcal{C}$ by [20] when considering $\mathcal{C}$ as a root-class.

From parts (i), (ii) and (iii), we conclude that $M' \in \mathcal{N}$.

(b) It remains to prove that $M' \cap \left(G_1 \underset{H_1}{*} G_2\right) \leq N$, i.e, $M' \cap \left(G_1 \underset{H_1}{*} G_2\right)$ is a subgroup of N.

Here, we use the presentation of groups by the generators and relations. Let,

$$G_i = \langle S_i | D_i \rangle, \quad H_i = \langle Q_i | V_i \rangle \quad \text{with } Q_i \subset S_i \cap S_{i+1},$$

$$G = \Big\langle \cup S_i | \cup D_i,\ f_{i1}(x) = f_{i2}(x)\ \forall\ x \in Q_i \Big\rangle$$

with $f_{i1} : H_i \to G_i$ and $f_{i2} : H_i \to G_{i+1}$ the embedding maps. Then

$$G_1 \underset{H_1}{*} G_2 = \Big\langle S_1 \cup S_2 | D_1 \cup D_2,\ f_{11}(x) = f_{12}(x)\ \forall\ x \in Q_1 \Big\rangle,$$

$$N = \langle A_1 \cup A_2 \cup A_3 | C \rangle \qquad \text{with } A_i \subset S_i,$$

$$M_1 = \langle I_1 \cup I_2 \cup I_3 | F \rangle \qquad \text{with } I_i \subset S_i,$$

$$M_2 = \langle J_1 \cup J_2 \cup J_3 | X \rangle \qquad \text{with } J_i \subset S_i,$$

$$M' = M_1 \cap M_2 = \Big\langle (I_1 \cup I_2 \cup I_3) \cap (J_1 \cup J_2 \cup J_3) | W \Big\rangle.$$

Since

$$M_1 \cap G_1 = N \cap G_1 \quad \Rightarrow \quad (\cup I_i) \cap S_1 = (\cup A_i) \cap S_1, \qquad i = 1, 2, 3$$

and

$$M_2 \cap G_2 = N \cap G_2 \quad \Rightarrow \quad (\cup J_i) \cap S_2 = (\cup A_i) \cap S_2, \qquad i = 1, 2, 3,$$

it follows that:

$$\begin{aligned} M' \cap \Big(G_1 \underset{H_1}{*} G_2\Big) &= M_1 \cap M_2 \cap \Big(G_1 \underset{H_1}{*} G_2\Big) \\ &= \Big\langle \big[(\cup I_i) \cap (\cup J_i)\big] \cap (S_1 \cup S_2) | Z \Big\rangle \\ &= \Big\langle \big[(\cup I_i) \cap (\cup J_i) \cap S_1\big] \cup \big[(\cup I_i) \cap (\cup J_i) \cap S_2\big] | Z \Big\rangle \\ &= \Big\langle \big[(\cup A_i) \cap S_1 \cap (\cup J_i)\big] \cup \big[(\cup I_i) \cap (\cup A_i) \cap S_2\big] | Z \Big\rangle \\ &= \Big\langle (\cup A_i) \cap \big[\big(S_1 \cap (\cup J_i)\big) \cup \big((\cup I_i) \cap S_2\big)\big] | Z \Big\rangle \\ &= \langle Y | Z \rangle \quad \text{with } Y \subset \cup A_i \\ &= A_1 \cup A_2 \cup A_3 \ \leq \ N. \end{aligned}$$

By (1) and (2) we conclude that $\mathcal{N}_1 = \mathcal{N}_{induced}$, i.e., the pro-$\mathcal{C}$ topology of G induces on $G_1 \underset{H_1}{*} G_2$ its own pro-$\mathcal{C}$ topology.

We argue similarly to prove that the pro-$\mathcal{C}$ topology of G induces on $G_2 \underset{H_2}{*} G_3$ his own pro-$\mathcal{C}$ topology. ∎

Proof of Theorem 1.2. Consider

$$\mathcal{N} = \big\{ N \lhd_f G : N \cap G_i \text{ is open in } G_i,\ i = 1,2,3 \text{ and } \ G/N \in \mathcal{C} \big\},$$

$$\mathcal{N}_1 = \Big\{ N \lhd_f G_1 \underset{H_1}{*} G_2 : N \cap G_i \text{ is open in } G_i,\ i = 1,2,\ \Big(G_1 \underset{H_1}{*} G_2\Big)/N \in \mathcal{C} \Big\},$$

$$\mathcal{N}_2 = \Big\{ N \lhd_f G_2 \underset{H_2}{*} G_3 : N \cap G_i \text{ is open in } G_i,\ i = 2,3,\ \Big(G_2 \underset{H_2}{*} G_3\Big)/N \in \mathcal{C} \Big\}.$$

We have:

$$G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3 = \overbrace{G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3}^{\ \mathcal{N}}, \tag{4.1}$$

$$\overbrace{G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3}^{\ \mathcal{N}} = \overbrace{\Big(G_1 \underset{H_1}{*} G_2\Big) \underset{G_2}{*} \Big(G_2 \underset{H_2}{*} G_3\Big)}^{\ \mathcal{N}}, \tag{4.2}$$

$$\overbrace{\Big(G_1 \underset{H_1}{*} G_2\Big) \underset{G_2}{*} \Big(G_2 \underset{H_2}{*} G_3\Big)}^{\ \mathcal{N}} = \widehat{G_1 \underset{H_1}{*} G_2}^{\mathcal{N}_1} \underset{\widehat{G_2}^{\mathcal{C}}}{\amalg} \widehat{G_2 \underset{H_2}{*} G_3}^{\mathcal{N}_2}, \tag{4.3}$$

$$\widehat{G_1 \underset{H_1}{*} G_2}^{\mathcal{N}_1} \underset{\widehat{G_2}^{\mathcal{C}}}{\amalg} \widehat{G_2 \underset{H_2}{*} G_3}^{\mathcal{N}_2} = \Big(G_1 \underset{H_1}{\amalg} G_2\Big) \underset{G_2}{\amalg} \Big(G_2 \underset{H_2}{\amalg} G_3\Big). \tag{4.4}$$

Equation (4.1) is the construction of free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamated subgroups (see Definition 2.1).

Equation (4.2) is obtained by writting the free abstract product of three abstract groups with amalgamated subgroups as a double amalgamation.

Equation (4.3) is obtained by [19] using two reasons:

1. $G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ induces on $G_1 \underset{H_1}{*} G_2$ and $G_2 \underset{H_2}{*} G_3$ their respective pro-$\mathcal{C}$ topologies (see Lemma 4.1), and
2. $G = G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ is $\mathcal{C}$-residual since G_1, G_2 and G_3 are $\mathcal{C}$-residual and the subgroups H_1 and H_2 are $\mathcal{C}$-separated (see Theorem 1.1).

Equation (4.4) is obtained by the construction of free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamation presented by L. Ribes and P. Zalesskii in [19], and using the equality $\widehat{G_2}^{\mathcal{C}} = G_2$ (by hypothesis). This completes the proof of the theorem. ∎

5. Proof of Theorem 1.3 and Corollary 1.4

Lemma 5.1. *Let $\mathcal{C}$ be the class of all finite solvable groups. Let G be a pro-$\mathcal{C}$ group. Let G_1, G_2, G_3, H_1 and H_2 be closed subgroups of G such that G_2 is generated by H_1 and H_2. Assume that the pro$-\mathcal{C}$ topology of G induces on G_1, G_2, G_3, H_1 and on H_2 their pro$-\mathcal{C}$ topologies. Assume also that H_1 is a common subgroup of G_1 and G_2, H_2 is a common subgroup of G_2 and G_3 such that H_1 is central in G_1, H_2 is abelian and commute with G_1, and H_1 and H_2 are $\mathcal{C}$-separable and $\widehat{G_2}^{\mathcal{C}} = G_2$. Then in $G = G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3$,*

$$G_1 \underset{H_1}{\amalg} G_2 = G_1 \amalg H_2 = (G_1 \amalg H_2) \underset{G_2}{\amalg} G_2.$$

Proof. It suffices to prove that in $G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$, we have

$$G_1 \underset{H_1}{*} G_2 = G_1 * H_2 = (G_1 * H_2) \underset{G_2}{*} G_2.$$

Indeed, assume that the above equality hold. Then the following sets are the same:

$$\mathcal{N}_a = \Big\{ N \lhd_f G_1 \underset{H_1}{*} G_2 : \, N \cap G_i \text{ is open in } G_i, i = 1, 2, \, \Big(G_1 \underset{H_1}{*} G_2 \Big) / N \in \mathcal{C} \Big\},$$

$$\mathcal{N}_b = \Big\{ N \lhd_f G_1 * H_2 : \begin{array}{c} N \cap G_1 \text{ is open in } G_1, \; N \cap H_2 \text{ is open in } H_2, \\ (G_1 * H_2)/N \in \mathcal{C} \end{array} \Big\},$$

$$\mathcal{N}_c = \left\{ N \lhd_f (G_1 * H_2) \underset{G_2}{*} G_2 : \begin{array}{c} N \cap G_i \text{ is open in } G_i,\ i = 1,2, \\ \Big((G_1 * H_2) \underset{G_2}{*} G_2\Big)/N \in \mathcal{C} \end{array} \right\}.$$

Consequently the following completions are equals:

$$\widehat{G_1 \underset{H_1}{*} G_2}^{\mathcal{N}_a} = \widehat{G_1 * H_2}^{\mathcal{N}_b} = \overbrace{(G_1 * H_2) \underset{G_2}{*} G_2}^{\mathcal{N}_c},$$

i.e., $G_1 \underset{H_1}{\amalg} G_2 = G_1 \amalg H_2 = (G_1 \amalg H_2) \underset{G_2}{\amalg} G_2$, as needed.

Let us now prove that in $G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$, we have $G_1 \underset{H_1}{*} G_2 = G_1 * H_2$.

Assume that for $i = 1,2$, $G_i = \langle S_i | D_i \rangle$, $H_i = \langle Q_i | D_i \rangle$ with $Q_1 \subset S_1$ and $S_2 = Q_1 \cup Q_2$.

$G_1 * H_2 = \big\langle S_1 \cup Q_2 | D_1 \cup D_2 \big\rangle$ and in $G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$, for all $x \in Q_1$, $f_{11}(x) = f_{12}(x)$. Then,

$$G_1 * H_2 = \big\langle S_1 \cup S_2 | D_1 \cup D_2,\ f_{11}(x) = f_{12}(x), \forall\, x \in Q_1 \big\rangle = G_1 \underset{H_1}{*} G_2$$

and the first equality has been proved.

• Let $\varphi_2 : G_2 \to G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$ be the inclusion map. Let $i_1 : G_2 \to G_1 * H_2$ and $i_2 : G_2 \to G_2$ be the corestrictions of φ_2 on $G_1 * H_2$ and G_2 respectively.

$G_1 * H_2 = \big\langle S_1 \cup Q_2 | D_1 \cup D_2 \big\rangle$ and in $G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} G_3$, $\forall x \in S_2$, $i_1(x) = i_2(x)$ and $S_2 = Q_1 \cup Q_2$. Then,

$$G_1 * H_2 = \big\langle S_1 \cup S_2 | D_1 \cup D_2,\ i_1(x) = i_2(x)\ \forall x \in S_2 \big\rangle = (G_1 * H_2) \underset{G_2}{*} G_2$$

and the second equality is proved. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. (i) ⇒ (ii) Assume that $G = G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3$. Since the conditions in Theorem 1.2 are satisfied, we write G as double pro-$\mathcal{C}$ amalgamation. That is:

$$G = G_1 \underset{H_1}{\amalg} G_2 \underset{H_2}{\amalg} G_3 = \Big(G_1 \underset{H_1}{\amalg} G_2\Big) \underset{G_2}{\amalg} \Big(G_2 \underset{H_2}{\amalg} G_3\Big).$$

Following Ribes and Zalesskii (see [19, Theorem 9.3.1]), it follows that the natural homomorphism

$$\begin{array}{rcl} \phi_G : \mathrm{Der}_{G_2}(G, A) & \longrightarrow & \mathrm{Der}_{G_2}\Big(G_1 \underset{H_1}{\amalg} G_2, A\Big) \times \mathrm{Der}_{G_2}\Big(G_2 \underset{H_2}{\amalg} G_3, A\Big) \\ f & \longmapsto & \Big(f|_{G_1 \underset{H_1}{\amalg} G_2}, f|_{G_2 \underset{H_2}{\amalg} G_3}\Big) \end{array}$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules $A \in \mathcal{C}$.

Now by Lemma 5.1, we have:

$$\mathrm{Der}_{G_2}\Big(G_1 \amalg_{H_1} G_2, A\Big) \times \mathrm{Der}_{G_2}\Big(G_2 \amalg_{H_2} G_3, A\Big) = \\ \mathrm{Der}_{G_2}\Big((G_1 \amalg H_2) \amalg_{G_2} G_2, A\Big) \times \mathrm{Der}_{G_2}\Big((G_3 \amalg H_1) \amalg_{G_2} G_2, A\Big).$$

Also, by [19, Theorem 9.3.1], the natural homomorphism

$$\phi^1_{(G_1 \amalg H_2) \amalg_{G_2} G_2} : \mathrm{Der}_{G_2}\Big((G_1 \amalg H_2) \amalg_{G_2} G_2, A\Big) \longrightarrow \\ \mathrm{Der}_{G_2}(G_1 \amalg H_2, A) \times \mathrm{Der}_{G_2}(G_2, A)$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules A $\in \mathcal{C}$.

Obviously, $\mathrm{Der}_{G_2}(G_2, A) = 0$. Now, let δ_1 be the isomorphism defined as

$$\delta_1 : \mathrm{Der}_{G_2}(G_1 \amalg H_2, A) \times 0 \longrightarrow \mathrm{Der}_{G_2}(G_1 \amalg H_2, A).$$

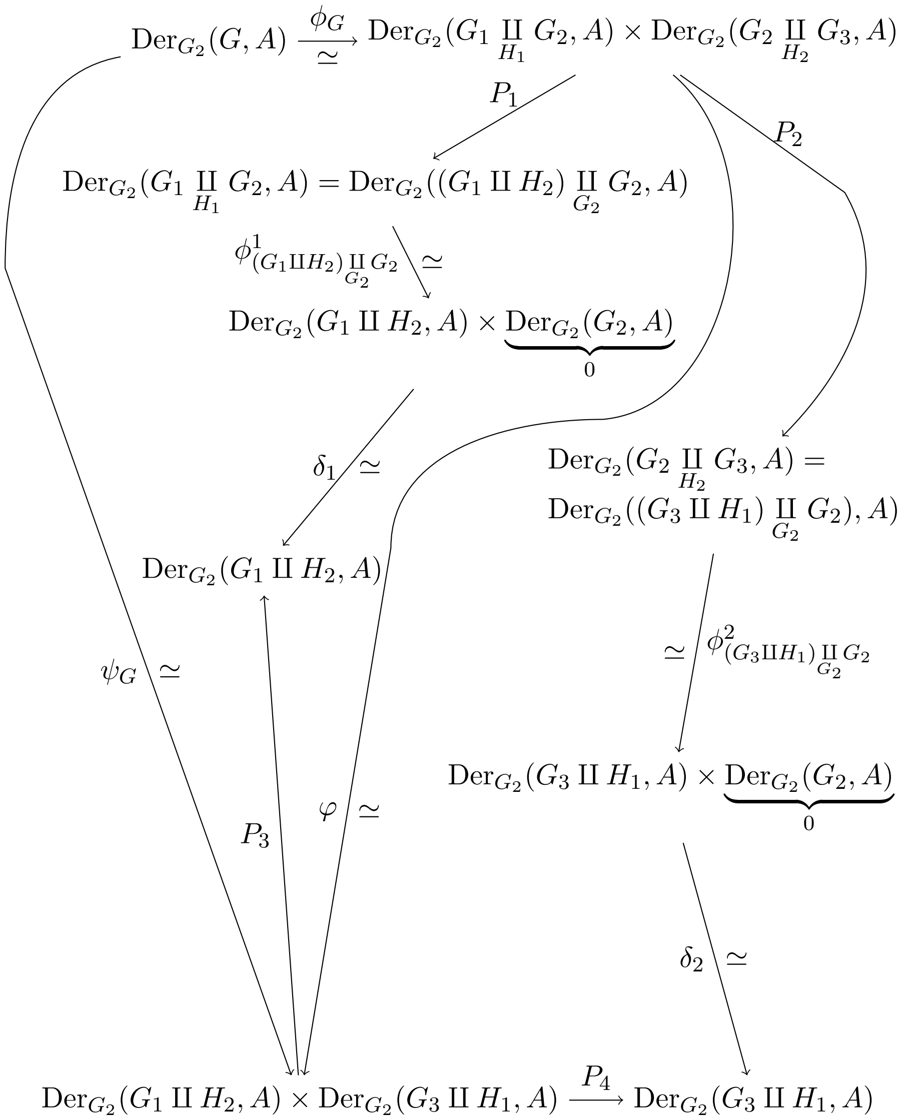
We obtain the isomorphism

$$\delta_1\phi^1_{(G_1 \amalg H_2) \amalg_{G_2} G_2} : \mathrm{Der}_{G_2}\Big(G_1 \amalg_{H_1} G_2, A\Big) \longrightarrow \mathrm{Der}_{G_2}(G_1 \amalg H_2, A).$$

Similarly, we obtain the isomorphism

$$\delta_2\phi^2_{(G_3 \amalg H_1) \amalg_{G_2} G_2} : \mathrm{Der}_{G_2}\Big(G_2 \amalg_{H_2} G_3, A\Big) \longrightarrow \mathrm{Der}_{G_2}(G_3 \amalg H_1, A).$$

The following diagram illustrates this situation.



where P_1, P_2, P_3 and P_4 are the canonical projections. We can see that

$$\mathrm{Der}_{G_2}\left(G_1 \underset{H_1}{\amalg} G_2, A\right) \times \mathrm{Der}_{G_2}\left(G_2 \underset{H_2}{\amalg} G_3, A\right),$$

$$\mathrm{Der}_{G_2}(G_1 \amalg H_2, A) \times \mathrm{Der}_{G_2}(G_3 \amalg H_1, A)$$

as direct products of the groups $\mathrm{Der}_{G_2}(G_1 \amalg H_2, A)$; $\mathrm{Der}_{G_2}(G_3 \amalg H_1, A)$. So, by the unicity of the direct product of two groups, $\psi_G = \varphi\phi_G$ is an isomorphism since so are φ and ϕ_G.

(ii) $\Rightarrow$ (i) Assume that the natural homomorphism

$$\begin{array}{rcl}\psi_G : \mathrm{Der}_{G_2}(G, A) & \longrightarrow & \mathrm{Der}_{G_2}(G_1 \amalg H_2, A) \times \mathrm{Der}_{G_2}(G_3 \amalg H_1, A) \\ f & \longmapsto & \left(f|_{G_1 \amalg H_2}, f|_{G_3 \amalg H_1}\right)\end{array}$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$ modules $A \in \mathcal{C}$. Using Lemma 5.1,

$$\phi_G : \ \mathrm{Der}_{G_2}(G, A) \longrightarrow \mathrm{Der}_{G_2}\left(G_1 \coprod_{H_1} G_2, A\right) \times \mathrm{Der}_{G_2}\left(G_2 \coprod_{H_2} G_3, A\right)$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules $A \in \mathcal{C}$. Applying ($2 \Rightarrow 1$) in [19, Theorem 9.3.1] we have:

$$G = \left(G_1 \coprod_{H_1} G_2\right) \coprod_{G_2} \left(G_2 \coprod_{H_2} G_3\right).$$

Since the conditions in Theorem 1.2 are satisfied, the following equality holds:

$$\left(G_1 \coprod_{H_1} G_2\right) \coprod_{G_2} \left(G_2 \coprod_{H_2} G_3\right) = G_1 \coprod_{H_1} G_2 \coprod_{H_2} G_3.$$

Thus, $G = G_1 \coprod_{H_1} G_2 \coprod_{H_2} G_3$. And the Theorem is proven. ∎

Proof of Corollary 1.4. Using the presentation of groups by generators and relators, we can write a free product of two groups with commuting subgroups as a free product of three groups with amalgamated subgroups. That is: $G_1 \underset{[H,K]}{*} G_2 = G_1 \underset{H}{*} (H \times K) \underset{K}{*} G_2$. Considering the construction of free profinite product of two profinite groups with commuting subgroups presented by G. Mantika and D. Tieudjo in [10], $G_1 \coprod_{[H,K]} G_2 = \widehat{G_1 \underset{[H,K]}{*} G_2}$. Consequently we have

$$G_1 \coprod_{[H,K]} G_2 = G_1 \coprod_{H} (H \times K) \coprod_{K} G_2.$$

The corollary follows immediately by applying Theorem 1.3 while taking into account Proposition 2.5. ∎

Acknowledgements

The authors would like to thank the referee for careful reading, constructive comments and helpful suggestions to improve the paper.

References

[1] G. Baumslag, On the residual finiteness of generalised free products of nilpotent groups, *Trans. Amer. Math. Soc.* **106** (1963), 193 – 209.

[2] N. Bourbaki, "General Topology", Springer-Verlag, Berlin, 1989.

[3] E. Detomi, B. Klopsch, P. Shumyatsky, Strong conciseness in profinite groups, *J. Lond. Math. Soc. (2)* **102** (2020), 977 – 993.

[4] S. Douboula, N. Temate, G. Mantika, D. Tieudjo, Multiple amalgamated free products and multiple free products with commuting subgroups by trees, *Int. J. App. Math.* **38:11s** (2025), 657 – 680.

[5] M. Ferrara, Join-distributive elements in the lattice of closed subgroups of a profinite group, *Adv. Group Theory Appl.* **18** (2024), 137 – 152.

[6] M. Ferrara, M. Trombetti, The pro-norm of a profinite group, *Israel J. Math.* **254** (2023), 399 – 429.

[7] D. Gildenhuys, E. Mackay, Triple cohomology and Galois cohomology of profinite groups, *Comm. Algebra* **1** (1974), 459 – 473.

[8] F. de Giovanni, I. de las Heras, M. Trombetti, On the lattice of closed subgroups of a profinite group, *Internat. J. Algebra Comput.* **34** (04) (2024), 515 – 542.

[9] E.D. Loginova, Residual finiteness of the free product of two groups with commuting subgroups, *Siberian Math. J.* **40** (2) (1999), 341 – 350.

[10] G. Mantika, D. Tieudjo, On the free profinite products of profinite groups with commuting subgroups, *Int. J. Group Theory* **5** (2) (2016), 25 – 40.

[11] B.H. Neumann, An essay on free products of groups with amalgamations, *Philos. Trans. Roy. Soc. London Ser. A* **246** (1954), 503 – 554.

[12] B.H. Neumann, Permutational products of groups, *J. Austral. Math. Soc.* **1** (1959/60), 299 – 310.

[13] H. Neumann, Generalized free products with amalgamated subgroups, *Amer. J. Math.* **70** (1948), 590 – 625.

[14] J. Neukirch, Freie Produkte pro-endlicher Gruppen und ihre Kohomologie, *Arch. Math. (Basel)* **22** (1971), 337 – 357.

[15] L. Ribes, Amalgamated products of profinite groups: Counterexamples, *Proc. Amer. Math. Soc.* **37** (1973), 413 – 416.

[16] L. Ribes, Cohomological characterization of amalgamated products of groups, *J. Pure Appl. Algebra* **4** (1974), 309 – 317.

[17] L. Ribes, On a cohomology theory for pairs of groups, *Proc. Amer. Math. Soc.* **21** (1969), 230 – 234.

[18] L. Ribes, "Profinite graphs and groups", Springer, Cham, 2017.

[19] L. Ribes, P.A. Zalesskii, "Profinite Groups", Springer-Verlag, Berlin, 2010.

[20] D. Tieudjo, Root-class residuality of some free construction, *JP J. Algebra Number Theory Appl.* **18** (2010), 125 – 143.