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Two miniatures relating the 2×2 minors of a square matrix and the symmetric group

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Abstract: In the first miniature we recall a puzzle proposed by Martin Gardner in 1957 and refreshed by Blasco in 2014. We prove some properties of the matrices involved. We show that Gardner magical matrices are tropically singular. In the second miniature we construct a counterexample to the following conjecture by R. Flores: for a square matrix $A = (a_{ij})$ of size n, if for each permutation σ there exists a 2×2 minor $\begin{pmatrix} a_{\sigma(k)k} & a_{\sigma(k)l} \\ a_{\sigma(l)k} & a_{\sigma(l)l} \end{pmatrix}$ of A which vanishes, then determinant of A vanishes.

Key words: determinant, rank, minor, symmetric group, variant of magic square.

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Standard magic squares are well–known mathematical puzzles known since antiquity. They admit different variants; one such variant is Martin Gardner's mathematical game of 1957 (cf. [2] by Gardner and [1] by Blasco). Neither Gardner nor Blasco give proofs. Quoting Gardner: "I have not been able to find out who first discovered this delightful version of the magic square, which is applicable to multiplication as well as to addition boxes." There are thirty seven medals (each showing some kind of magic square on one side) in the National Archeological Museum (Madrid, Spain) all of them cast in metal. Apparently, they were carried as amulets, (cf. [5]).

Let $\mathbb{K}$ be a field. For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$, let S_n denote the *symmetric group* in n indices and consider a square matrix $A = (a_{ij}) \in M_n(\mathbb{K})$. Let $\operatorname{rk}(A)$ denote the *rank* of A. For each $\sigma \in S_n$, let

$$p(\sigma) := a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \in \mathbb{K}.$$

The *Leibniz formula* for the *determinant* of A reads

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) p(\sigma). \tag{0.1}$$

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We consider the *additive version* of the product $p(\sigma)$, namely

$$a(\sigma) := a_{\sigma(1)1} + a_{\sigma(2)2} + \cdots + a_{\sigma(n)n} \in \mathbb{K}.$$

The first miniature below brings together four items: (a) the family $a(\sigma)$, with $\sigma \in S_n$, (b) Gardner's new kind of magic square, (c) the family of 2×2 minors of A, and (d) the condition $\mathrm{rk}(A) \leq 2$. Our approach adds new features to Gardner's.

The second miniature below brings together two items: (α) the vanishing, for each $\sigma \in S_n$, of one 2×2 minor of A and (β) the vanishing of $\det(A)$. Ramón Flores from U. Sevilla, (Spain) conjectured (private communication) that (α) implies (β). The idea behind Flores' conjecture is that a matrix satisfying (α) apparently has many 2×2 vanishing minors. We give a counterexample to Flores' conjecture below.

We call them miniatures, following one inspiring book by Jiři Matoušek (cf. [3].)

1. Miniature 1

The matrices given by Gardner and Blasco are

$$A = \begin{pmatrix} 19 & 8 & 11 & 25 & 7 \\ 12 & 1 & 4 & 18 & 0 \\ 16 & 5 & 8 & 22 & 4 \\ 21 & 10 & 13 & 27 & 9 \\ 14 & 3 & 6 & 20 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}, \begin{pmatrix} 28 & 26 & 30 & 27 & 29 & 25 \\ 34 & 32 & 36 & 33 & 35 & 31 \\ 16 & 14 & 18 & 15 & 17 & 13 \\ 4 & 2 & 6 & 3 & 5 & 1 \\ 10 & 8 & 12 & 9 & 11 & 7 \\ 22 & 20 & 24 & 21 & 23 & 19 \end{pmatrix}.$$

With a different wording, the puzzle proposed by Gardner and Blasco asks to check that for each matrix A above, there exists a number $s(A) \in \mathbb{K}$ such that $a(\sigma) = s(A)$, for all $\sigma \in S_n$. We call *Gardner magical* any matrix A for which such and $s(A)$ exists.

Notice that summing over all $\sigma \in S_n$, we get $(n-1)! \sum_{i,j=1}^n a_{ij} = n! s(A)$, whence

$$\sum_{i,j=1}^{n} a_{ij} = n\, s(A).$$

Let G_n denote the set of all Gardner magical matrices of size n with entries in $\mathbb{K}$. The set G_n is a vector subspace of $M_n(\mathbb{K})$ described by a family of $n!$

homogeneous linear equations in n^2 indeterminates. Let us look closer at the cases $n = 2$ and 3.

If $n = 2$, then $a_{11} + a_{22} = a_{12} + a_{21}$ is the only equation describing G_2 and the condition Gardener magical is equivalent to $\mathrm{rk}(A) < 2$. If $n = 3$, then G_3 is described by six equations

$$\left.\begin{aligned} a_{11} + a_{22} + a_{33} &= s(A) \\ a_{21} + a_{32} + a_{13} &= s(A) \\ a_{31} + a_{12} + a_{23} &= s(A) \\ a_{31} + a_{22} + a_{13} &= s(A) \\ a_{21} + a_{12} + a_{33} &= s(A) \\ a_{11} + a_{32} + a_{23} &= s(A) \end{aligned}\right\}.$$

Subtracting the first from the last equations, we get that $A \in G_3$ implies that $A_{11} \in G_2$, where A_{ij} denotes the $(n-1) \times (n-1)$ submatrix obtained by omitting the i–th row and j–th column in A, $i, j \in [n]$. Other subtractions yield $A_{ij} \in G_2$, for all $i, j \in [3]$. The converse is easy to prove: if A_{ij} is Gardner magical for all $i, j \in [3]$, then $A \in M_3(\mathbb{K})$ is Gardner magical.

For $n \geq 4$ we have $n! > n^2$ and so the linear system describing the subspace G_n is, apparently, overdetermined: it has more equations than indeterminates and might have only the zero solution. However, the method of descending in the dimension shown for $n = 3$ works in general: $A \in G_n$ if and only if each 2×2 minor of A belongs to G_{n-1}. In symbols, G_n is given by the linear system

$$\Sigma_n : a_{ii} + a_{jj} = a_{ij} + a_{ji},\ 1 \leq i < j \leq n.$$

There are more equations in Σ_n than indeterminates, so Σ_n might have only the zero solution. However, this is not so.

Theorem 1.1. *The dimension of G_n is $2n - 1$.*

Proof. Fix one row index $i_0 \in [n]$ and one column index $j_0 \in [n]$. Once the entries $a_{i_0 j}$ and $a_{i j_0}$ are freely filled in a matrix $A = (a_{ij})$, then the rest of the entries are uniquely determined by the system of equations Σ_n, in order for A to be Gardner magical. It follows that the dimension of G_n is $n + (n-1) = 2n - 1$. (Alternative proof: fix one row (resp. column) index $i_0 \in [n]$ and freely fill in a matrix $A = (a_{ij})$ the entries $a_{i_0 j}$ (resp. $a_{j i_0}$) as well as the diagonal entries.) ∎

The following theorem is Gardner's explanation to the puzzle. Let $J_{n\times m}$ be the all–ones matrix of size $n \times m$.

Theorem 1.2. ([2]) *$A = (a_{ij})$ is Gardner magical if and only if the exist x_i and y_j such that $a_{ij} = x_i + y_j$, for $i, j \in [n]$. In symbols, $A = (x_1, x_2, \cdots, x_n)^T J_{1\times n} + J_{n\times 1}(y_1, y_2, \cdots, y_n)$.*

Proof. Assume $a_{ij} = x_i + y_j$. Then $a(\sigma) = \sum_{j=1}^n (x_{\sigma(j)} + y_j) = \sum_{j=1}^n x_j + \sum_{j=1}^n y_j$ does not depend on σ. For the converse, assume A is Gardner magical, choose any $k \in \mathbb{K}$, set $x_i = a_{i1} - a_{11} + k$ and $y_j = a_{1j} - k$. Then $a_{ij} = -a_{11} + a_{i1} + a_{1j} = x_i + y_j$. ∎

Corollary 1.3. *If $A \in G_n$, then* $\mathrm{rk}(A) \leq 2$.

Proof. If A is Gardner magical, then A is the sum of two matrices, each summand being the product of a column matrix and a row matrix. The rank of each summand is less than or equal to 1, and so the rank of A is less than or equal to 2. ∎

Notice that $\mathrm{rk}(A) \leq 2$ does not imply that $A \in G_n$, because the set of matrices of rank less that or equal to 2 is not a vector subspace, but G_n is a vector subspace.

Examples 1.4. (a) If A is Gardner magical, then $E_{ij}A$, AE_{kl}, $E_{ij}AE_{kl}$ and A^T are Gardner magical, where E_{ij} is a permutation matrix. This is true since the families of 2×2 minors of all these matrices are the same.

(b) If F is a row matrix, then the rows of the matrix $J_{n\times 1}F$ are all equal, and thus $J_{n\times 1}F$ is Gardner magical.

(c) If A is Gardner magical and has vanishing diagonal, then A is skew–symmetric.

2. Miniature 2

Given a matrix $A \in M_n(\mathbb{K})$, let $A^{ij,kl}$ denote the 2×2 submatrix obtained from rows i–th and j–th and columns k–th and l–th in A, for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$.

Definition 2.1. If $\det A^{ij,kl} = 0$, we say that the minor $A^{ij,kl}$ *vanishes.*

Definition 2.2. We say that $\sigma \in S_n$ *enters* $A^{ij,kl}$ if two entries of $A^{ij,kl}$ appear in $p(\sigma)$.

Notice that σ enters $A^{ij,kl}$ means that $\sigma(k) = i$ and $\sigma(l) = j$ or $\sigma(k) = j$ and $\sigma(l) = i$ whence, equivalently, $\sigma(\{k, l\}) = \{i, j\}$.

CONJECTURE 2.3. R. Flores: If for each $\sigma \in S_n$ there exist $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ such that σ enters the vanishing minor $A^{ij,kl}$, then $\det(A) = 0$.

At first sight, the hypothesis in Flores' conjecture guarantees a large number of vanishing 2×2 minors in A. A finer look gives the following.

LEMMA 2.4. *If σ enters $A^{ij,kl}$, then $(ij)\sigma$ enters $A^{ij,kl}$. In addition, if $A^{ij,kl}$ vanishes, then $p(\sigma) = p((ij)\sigma)$.*

Proof. Consider $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ and, without loss of generality, assume $\sigma(k) = i$ and $\sigma(l) = j$. Write $\tau = (ij)\sigma$. The boxed entries show the first assertion

$$\begin{pmatrix} \boxed{a_{\sigma(k)k}} & a_{\sigma(k)l} \\ a_{\sigma(l)k} & \boxed{a_{\sigma(l)l}} \end{pmatrix} = A^{ij,kl} = \begin{pmatrix} a_{\tau(l)k} & \boxed{a_{\tau(l)l}} \\ \boxed{a_{\tau(k)k}} & a_{\tau(k)l} \end{pmatrix}.$$

Besides, if $A^{ij,kl}$ vanishes, then $\boxed{a_{\sigma(k)k}a_{\sigma(l)l}} = a_{\sigma(l)k}a_{\sigma(k)l} = \boxed{a_{\tau(k)k}a_{\tau(l)l}}$. Further, $\sigma(r) = \tau(r)$, for all $r \in [n] \setminus \{k, l\}$, because

$$\tau(r) = (ij)\sigma(r) = \begin{cases} j & \text{if } r = k, \\ i & \text{if } r = l, \\ \sigma(r) & \text{otherwise.} \end{cases}$$

This implies $p(\sigma) = p(\tau)$. ∎

Notice that if the hypothesis of Flores' conjecture holds, then for each $\sigma \in S_n$ and each pair of indices $i < j$ such that σ enters the vanishing minor $A^{ij,kl}$, then the terms in Leibniz's formula (0.1) corresponding to σ and $(ij)\sigma$ cancel each other. If $\sigma \mapsto (ij)\sigma$ were a bijection in S_n, we would conclude $\det(A) = 0$. But this is not a bijection. Our counterexample to Flores' conjecture uses a *perturbed Bose–Mesner matrix* (cf. [4] for the Bose–Mesner matrix). It is a rank–one perturbation of a diagonal matrix.

Let I_n be the identity and J_n be the all–ones square matrix of size n.

DEFINITION 2.5. The *perturbed Bose–Mesner* matrix is $P_n(h) = (a_{ij})$,

$$a_{ij} = \begin{cases} h & \text{if } i = j \equiv 1(2), \\ h^{-1} & \text{if } i = j \equiv 0(2), \\ 1 & \text{otherwise.} \end{cases}$$

For instance

$$P_7(h) = \begin{pmatrix} h & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & h^{-1} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & h & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & h^{-1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & h & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & h^{-1} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & h \end{pmatrix}.$$

LEMMA 2.6. *If* $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$, *then* $\det(D+aJ_n) = d_1 d_2 \cdots d_n + a\sum_{i=1}^n d_1 \cdots d_{i-1} d_{i+1} \cdots d_n$, *where* $a \in \mathbb{K}$.

Proof. We use the following well–known *property of determinants*: $\det(A) = \det(A') + \det(A'')$, whenever there exists $i \in [n]$ such that the i–th row in A is the sum of the i–th row in A' and the i–th row in A'', while the j–th rows in A, A', A'' are equal, for all $j \in [n]$, $j \neq i$.

Take $A' = D$, $A'' = aJ_n$ and apply the former property to all the rows both in D and aJ_n, obtaining $\det(A)$ decomposed into 2^n summands, among which all those including two rows from aJ_n vanish. The remaining summands add up to $\det(D) + a\sum_{i=1}^n d_1 \cdots d_{i-1} d_{i+1} \cdots d_n$, and therefore, this is the value of $\det(D + aJ_n)$. ∎

COROLLARY 2.7. *For* $n \geq 1$ *we have*

$$\det P_n(h) = \begin{cases} -m(2-h-h^{-1})^{m+1} & \text{if } n = 2m+2, \\ p(h,m)(2-h-h^{-1})^{m-1} & \text{if } n = 2m+1, \end{cases}$$

where $p(h,m) = m(h^2 - 3h + 3 - h^{-1}) + h(2 - h - h^{-1})$.

Proof. $P_n(h) = D + J_n$, where $D = \operatorname{diag}(h-1, h^{-1}-1, h-1, h^{-1}-1, \dots)$. By Lemma 2.6, $\det P_n(h) = \det(D) + t$, where $t = \sum_{i=1}^n d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_n$. Clearly,

$$\det(D) = \begin{cases} (h-1)^{m+1}(h^{-1}-1)^{m+1} = (2-h-h^{-1})^{m+1} & \text{if } n = 2m+2, \\ (2-h-h^{-1})^m (h-1) & \text{if } n = 2m+1, \end{cases}$$

with $m \in \mathbb{N} \cup \{0\}$. Now t is a sum of n terms, each term is a product of $n-1$ factors, each factor being equal to $h-1$ or $h^{-1}-1$. Easy counting yields

$$t = \begin{cases} -(m+1)(2-h-h^{-1})^{m+1} & \text{if } n = 2m+2, \\ (m+1)(2-h-h^{-1})^m + m(2-h-h^{-1})^{m-1}(h-1)^2 & \text{if } n = 2m+1. \end{cases}$$

Adding up and gathering terms, we get the result. ∎

Remark 2.8. The 2×2 minors in $P_n(h)$ are:

1. vanishing: $A^{ij,kl} = \begin{pmatrix} h & 1 \\ 1 & h^{-1} \end{pmatrix}, \begin{pmatrix} h^{-1} & 1 \\ 1 & h \end{pmatrix}$, whenever $i \not\equiv j(2)$ and $(i,j) = (k,l)$ (central minor);
2. vanishing: $A^{ij,kl} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, whenever $\{i,j\} \cap \{k,l\} = \emptyset$ (peripheral minor);
3. the rest are non–vanishing minors, if $h \neq \pm 1$: these are

$$\begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} h^{-1} & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & h \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & h^{-1} \end{pmatrix}, \begin{pmatrix} h & 1 \\ 1 & h \end{pmatrix}, \begin{pmatrix} h^{-1} & 1 \\ 1 & h^{-1} \end{pmatrix}.$$

NOTATION 2.9. For $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, consider the sets

$$\begin{aligned} S^{A,ij,kl,+} &:= \{\sigma \in S_n : \det\left(A^{ij,kl}\right) = 0,\ \sigma(k) = i,\ \sigma(l) = j\}, \\ S^{A,ij,kl,-} &:= \{\sigma \in S_n : \det\left(A^{ij,kl}\right) = 0,\ \sigma(k) = j,\ \sigma(l) = i\}, \\ S^{A,ij,kl} &:= S^{A,ij,kl,+} \cup S^{A,ij,kl,-}. \end{aligned}$$

The hypothesis in Flores' conjecture is equivalent to the inclusion

$$S_n \subseteq \bigcup_{1 \leq i < j \leq n,\quad 1 \leq k < l \leq n} S^{A,ij,kl}. \tag{2.1}$$

EXAMPLE 2.10. Consider $A = P_n(h)$ and choose $h \in \mathbb{K}$ such that $\det P_n(h) \neq 0$ (cf. Corollary 2.7; in particular, $h \neq 1$). Next we show (2.1), whenever $n \gg 0$, providing a counterexample to Flores' conjecture.

We will list several mutually exclusive cases covering all $\sigma \in S_n$.

CASE 1: IF $\sigma \in S_n$ FIXES, AT LEAST, TWO INDICES OF DIFFERENT PARITY. Say $\sigma(i) = i$ and $\sigma(j) = j$ with $i \not\equiv j(2)$, $i, j \in [n]$, $i < j$. Then $\sigma \in S^{A,ij,ij}$, i.e., σ enters a vanishing central minor. This case includes $\sigma = \mathrm{id}$, for $n \geq 2$.

CASE 2: IF $\sigma \in S_n$ FIXES AT MOST ONE INDEX, OR σ FIXES AT LEAST TWO INDICES, BUT ALL THE INDICES FIXED BY σ HAVE THE SAME PARITY. Since $n \gg 0$, there are at least FOUR different indices NOT fixed by σ.

Recall that $\mathrm{id} \neq \sigma \in S_n$ factors uniquely into a finite product of non–trivial disjoint cycles. Furthermore, disjoint cycles commute.

Case 2a: If the length of the longest cycle dividing σ is larger than 3. Let $c = (p, \sigma(p), \sigma^2(p), \sigma^3(p), \ldots)$ be one of the longest cycles dividing σ. Then $\sigma \in S^{A,mM,m'M'}$ with $m = \min\{\sigma(p), \sigma^3(p)\}$, $M = \max\{\sigma(p), \sigma^3(p)\}$, $m' = \min\{p, \sigma^2(p)\}$, $M' = \max\{p, \sigma^2(p)\}$, and $\{m, M\} \cap \{m', M'\} = \emptyset$, i.e., σ enters a vanishing peripheral minor $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Case 2b: If the length of the longest cycle dividing σ is no larger than 3. Then σ has at least two disjoint cyclic factors, each one of length at least 2. Let $c_1 = (p, \sigma(p), \ldots)$ and $c_2 = (q, \sigma(q), \ldots)$, $c_1 \neq c_2$ be two such cycles dividing σ. Then $\sigma \in S^{A,mM,m'M'}$ with $m = \min\{\sigma(p), \sigma(q)\}$, $M = \max\{\sigma(p), \sigma(q)\}$, $m' = \min\{p, q\}$, $M' = \max\{p, q\}$, and $\{m, M\} \cap \{m', M'\} = \emptyset$, i.e., σ enters a vanishing peripheral minor $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

We are done!

From the proof, we see that $\lceil \frac{n}{2} \rceil + 4 \leq n$, i.e., $n \geq 8$ suffices.

Appendix

The *permanent* of A is the *determinant without signs*, defined as

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} p(\sigma). \tag{A.1}$$

One common version of *tropical mathematics* on an ordered algebraic structure (for instance, an ordered field $\mathbb{K}$), is mathematics with product replaced by sum, and sum replaced by maximum. Assuming that the field $\mathbb{K}$ is ordered, the *tropical version* of (A.1) is

$$\mathrm{troper}(A) = \max_{\sigma \in S_n} a(\sigma).$$

A Gardner magical square A attains the maximum $\mathrm{troper}(A)$ at any permutation $\sigma \in S_n$. By definition, a square matrix A is *tropically singular* if the maximum $\mathrm{troper}(A)$ is attained, at least, at two different permutations $\sigma \in S_n$. Therefore Gardner magical is stronger than tropically singular.

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