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Binet's formula for operator-valued recursive sequences and the operator moment problem

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Abstract: We derive a Binet-type formula for operator-valued sequences satisfying linear recurrence relations, extending the classical scalar case to the setting of bounded operators on Hilbert spaces. In this framework, we analyze the operator moment problem as an application, establishing new connections between recursive operator sequences and moment sequences.

Key words: Binet formula, recursive operator-valued sequences, operator moment problem, representing measures, flat extension.

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1. Introduction

Let r be an integer such that $r \geq 2$. Consider $\{\gamma_n\}_{n\geq 0}$ a sequence of complex numbers given by

$$\begin{cases} \gamma_{n+1} = a_0\gamma_n + a_1\gamma_{n-1} + \cdots + a_{r-1}\gamma_{n-r+1} \qquad \text{for all } n \geq r-1, \\ \quad \gamma_0 = \alpha_0, \quad \gamma_1 = \alpha_1, \quad \ldots, \quad \gamma_{r-1} = \alpha_{r-1}, \end{cases} \tag{1.1}$$

where $a_0, a_1, \ldots, a_{r-1}$ (with $a_{r-1} \neq 0$) and $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ are given complex numbers.

Sequences satisfying (1.1), known as r-generalized Fibonacci sequences, appear in various fields of mathematics and computer science and have been studied using different methods (see [12, 14, 15, 8]).

The polynomial function $P \in \mathbb{C}[X]$ given by

$$P(X) = X^r - a_0X^{r-1} - \cdots - a_{r-2}X - a_{r-1} \tag{1.2}$$

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is called the characteristic polynomial of the sequence (1.1). Let $\lambda_1, \ldots, \lambda_s$ be its roots with respective multiplicities $m_1, \ldots, m_s$.

The classical Binet formula for the solution of (1.1) is:

$$\gamma_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j} n^j \right) \lambda_i^n, \qquad n \geq 0, \tag{1.3}$$

where the coefficients $\beta_{i,j}$ are uniquely determined by the initial conditions $\{\alpha_j\}_{0 \leq j \leq r-1}$ and satisfy the $r \times r$ system (see [9, 11, 5]):

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j} k^j \right) \lambda_i^k = \alpha_k, \qquad k = 0, 1, \ldots, r-1.$$

In the case when the sequence (γ_n) is substituted by a given operator-valued sequence $\mathcal{T} = (T_n)_{n \in \mathbb{Z}_+}$ of bounded operators on a Hilbert space $\mathcal{H}$, we say that $\mathcal{T}$ is a r-generalized operator-valued Fibonacci sequence. That is $\mathcal{T} = (T_n)_{n \in \mathbb{Z}_+}$ satisfies a linear recurrence relation of the form

$$T_{n+1} = a_0 T_n + a_1 T_{n-1} + \cdots + a_r T_{n-r} \qquad \text{for all } n \geq r, \tag{1.4}$$

where $a_0, \ldots, a_r \in \mathbb{R}$ are fixed coefficients and $T_0, \ldots, T_{r-1}$ are given bounded operators on $\mathcal{H}$.

The study of such sequences is motivated by the natural progression from scalar to matrix and operator-valued recurrences, which arise in diverse areas, including: Functional analysis (operator dynamics [6], semigroup theory [4]), Dynamical systems (evolution equations [10], stability analysis [20]), Numerical analysis (iterative methods for operator equations [21]), Computer science (algorithmic complexity [2], quantum computing [7]), among others.

These sequences are analyzed using techniques from spectral theory, matrix analysis, and noncommutative algebra, making them relevant to both theoretical and applied mathematics.

Throughout this paper, we use the standard notations: $\mathbb{C}$ for complex numbers, $\mathbb{R}$ for real numbers, $\mathbb{N} = \{1, 2, \ldots\}$ for positive integers, and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for non-negative integers. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $\mathbb{K}[X]$ denotes the algebra of polynomials with coefficients in $\mathbb{K}$. $\mathcal{Z}(P)$ denotes the set of zeros of a polynomial $P \in \mathbb{K}[X]$. We also denote $\mathcal{H}$ and $\mathcal{K}$ separable complex Hilbert spaces and $\mathbf{B}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$, T^*, and $\sigma(T)$ denote the kernel, the range, the adjoint and the spectrum of T respectively. The space $\mathbf{B}(\mathcal{H}) := \mathbf{B}(\mathcal{H},\mathcal{H})$ is a unital C^*-algebra, with identity $I_{\mathcal{H}}$ and null operator $0_{\mathcal{H}}$.

We denote the inner product and the norm on $\mathcal{H}$ by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively (or simply $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ when unambiguous). The unit sphere in $\mathcal{H}$ is the set of normalized vectors in $\mathcal{H}$, which is given by $\mathcal{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| = 1\}$.

An operator $T \in \mathbf{B}(\mathcal{H})$ is self-adjoint if $T = T^*$, positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and an orthogonal projection if $T = T^*$ and $T^2 = T$. We write $T \geq 0$ for a positive operator T.

The real vector space of all self-adjoint bounded operators is denoted by $\mathbf{B}_h(\mathcal{H})$, and the cone of all positive operators by $\mathbf{B}_+(\mathcal{H})$. The space of all sequences $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+}$ with $T_n \in \mathbf{B}_h(\mathcal{H})$ for all n is denoted $\mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$.

An operator-valued charge (OVC) is a mapping $E : \mathcal{B}(\mathbb{R}) \to \mathbf{B}(\mathcal{H})$ ($\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$) which satisfies the condition that for all $x, y \in \mathcal{H}$, the function $E_{x,y}(\cdot) := \langle E(\cdot)x, y\rangle$ is a complex Borel measure on $\mathbb{R}$. Furthermore:

- E is an operator-valued measure (OVM) if $E_x(\cdot) := \langle E(\cdot)x, x\rangle$ is a positive measure for all $x \in \mathcal{H}$;
- E is a semi-spectral measure if it is an OVM and $E(\mathbb{R}) = I_{\mathcal{H}}$;
- E is a spectral measure if it is semi-spectral and $E(\mathcal{B}(\mathbb{R}))$ is a subset of the set of all orthogonal projections defined on $\mathcal{H}$.

The following result describes the relationship between OVMs and spectral measures.

Theorem 1.1. (Naimark's Dilation Theorem [13, Theorem 4]) *Let $E : \mathcal{B}(\mathbb{R}) \to \mathbf{B}_+(\mathcal{H})$ be an operator-valued measure. Then there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$, and a spectral measure $F : \mathcal{B}(\mathbb{R}) \to \mathbf{B}_+(\mathcal{K})$ such that*

$$E(\cdot) = V^*F(\cdot)V.$$

Moreover, if E is a semi-spectral measure, then $\mathcal{H} \subseteq \mathcal{K}$ and $E(\cdot) = P_{\mathcal{H}}F(\cdot)$ on $\mathcal{H}$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

A recent criterion to determine when a *semi-spectral measure* is *spectral* has been obtained by P. Pietrzycki and J. Stochel in [16, Theorem 4.2] and [17]. According to Naimark's Theorem, the integration theory for OVMs can be represented as an extension of spectral measures.

Consider an operator-valued charge (OVC) E, we define the support of E as the smallest closed subset A of $\mathbb{R}$ such that $E(\mathbb{R} \setminus B) = 0_{\mathcal{H}}$ for every Borel set $B \supseteq A$. (For a related definition, see [18, p. 69] or [1, Definition 16], in which the author designates the support of E as the co-spectrum of E.

An OVC with finite support is said to be finitely atomic. This is a very interesting class of operator-valued charges. It will be represented in the following form:

$$E = \sum_{k=1}^{r} S_k \delta_{\lambda_k}, \tag{1.5}$$

where $\operatorname{supp}(E) = \{\lambda_1, \lambda_2, \dots, \lambda_r\} \subseteq \mathbb{R}$, and $S_1, S_2, \dots, S_r \in \mathbf{B}(\mathcal{H})$. It is clear that a finitely atomic measure of the form (1.5) is an OVM if and only if $S_1, S_2, \dots, S_r \in \mathbf{B}_+(\mathcal{H})$.

Let E be an OVC supported on a closed subset K of $\mathbb{R}$. For $n \in \mathbb{Z}_+$, the integral of the monomial t^n with respect to $dE(t)$, denoted $\int_K t^n dE(t) \in \mathbf{B}(\mathcal{H})$, is defined by:

$$\left\langle \int_K t^n dE(t)x,\ y \right\rangle_{\mathcal{H}} = \int_K t^n \langle dE(t)x, y\rangle_{\mathcal{H}} \qquad \text{for all } x, y \in \mathcal{H},$$

provided all integrals on the right-hand side converge. This is called the n^{th} operator moment of E.

In the case where E is finitely atomic, given by (1.5), the n^{th} operator moment is:

$$T_n = \int_K t^n dE(t) = \sum_{k=1}^{r} \lambda_k^n S_k.$$

In the literature, for a sequence $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$, the term "operator moment sequence on K" typically refers to the existence of integral representations in the operator moment form (1.6) below, as in the following definition.

Definition 1.2. A sequence $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ is called an operator K–moment sequence, where $K \subseteq \mathbb{R}$ is closed, if there exists an operator-valued measure E supported on K, such that

$$T_n = \int_K t^n dE(t). \tag{1.6}$$

In this case, E is a representing operator-valued measure for the operator moment sequence $\mathcal{T}$.

In what follows, we present a Binet-type formulation for linear recurrence relations in the algebra of bounded operators on a complex Hilbert space $\mathcal{H}$, and apply it to solve the truncated operator moment problem on a finite subset of $\mathbb{R}$.

PROBLEM 1.3. (OPERATOR MOMENT PROBLEM) Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ and let K be a closed subset of $\mathbb{R}$. Find necessary and sufficient conditions to ensure that $\mathcal{T}$ is an operator K–moment sequence.

We refer to [3] for a comprehensive discussion of the operator moment problem.

PROBLEM 1.4. (BINET'S FORMULA FOR OPERATOR-VALUED RECURRENCES) Let $\mathcal{H}$ be a Hilbert space and $\mathbf{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

Consider a sequence $(T_n)_{n\geq 0} \subset \mathbf{B}(\mathcal{H})$ satisfying a linear recurrence relation of order r:

$$T_{n+r} = A_{r-1}T_{n+r-1} + \cdots + A_0 T_n,$$

where $A_i \in \mathbf{B}(\mathcal{H})$.

Does there exist an explicit Binet-type formula expressing T_n in terms of the spectral properties of the operator coefficients $\{A_i\}$?

In the first section, we provide an overview of essential concepts and properties that form the foundation of our theory. This includes a detailed discussion of operator-valued sequences, their definitions, and the frameworks required to understand the implications of Binet-type formulas.

In the second section, we extend the classical Binet formula to accommodate operator-valued sequences defined by linear recurrence relations. Through this generalization, we construct an explicit representation that not only mimics the traditional Binet formula for scalar sequences but also adapts it to the context of bounded operators on Hilbert spaces.

Finally, in the third section, we leverage the generalized Binet formula to characterize specific operator moment sequences. By applying this formula, we explore various properties of the sequences, ultimately contributing to the understanding of the operator moment problem. This section highlights examples and applications, showcasing the relevance of our findings in relation to both theoretical and practical scenarios within the field.

2. Main results

2.1. Binet formula for a linear recursive operator-valued sequence with scalar coefficients. We begin by considering the algebra endomorphism τ on $\mathbf{B}(\mathcal{H})^{\mathbb{Z}_+}$ defined by

$$\tau\left((T_n)_{n\in\mathbb{Z}_+}\right) := (T_{n+1})_{n\in\mathbb{Z}_+}.$$

For each $k \in \mathbb{Z}_+$, its k^{th} iterate is

$$\tau^k\left((T_n)_{n\in\mathbb{Z}_+}\right) = (T_{n+k})_{n\in\mathbb{Z}_+}.$$

Given a polynomial $P(X) = \sum_{k=0}^{r} a_k X^k \in \mathbb{R}[X]$, we define $P(\tau)$ by

$$P(\tau)\left((T_n)_{n\in\mathbb{Z}_+}\right) := \left(\sum_{k=0}^{r} a_k T_{n+k}\right)_{n\in\mathbb{Z}_+}.$$

We now introduce an alternative method for defining recursive operator-valued sequences, see [3].

Definition 2.1. A sequence $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}(\mathcal{H})^{\mathbb{Z}_+}$ is called a *linear recursive sequence* (LRS) if there exists a non-zero polynomial $P \in \mathbb{R}[X]$ such that $P(\tau)(\mathcal{T}) = (0_{\mathcal{H}})_{n\in\mathbb{Z}_+}$. Such a polynomial P is called a *characteristic polynomial* associated with $\mathcal{T}$.

By convention, the zero polynomial is a characteristic polynomial of every operator sequence.

Let $\mathcal{T}$ be a linear recursive sequence and define the algebra homomorphism

$$\Psi_{\mathcal{T}} : \mathbb{R}[X] \longrightarrow \mathbf{B}(\mathcal{H})^{\mathbb{Z}_+}, \qquad P \longmapsto P(\tau)(\mathcal{T}).$$

The kernel $\mathcal{P}(\mathcal{T}) := \ker \Psi_{\mathcal{T}} \subseteq \mathbb{R}[X]$ consists of all characteristic polynomials of $\mathcal{T}$. Since $\mathbb{R}[X]$ is a principal ideal domain, $\mathcal{P}(\mathcal{T})$ is generated by a unique monic polynomial $P_{\mathcal{T}}$ of minimal degree. We call:

- $P_{\mathcal{T}}$ the *minimal characteristic polynomial* of $\mathcal{T}$;
- the equation $P_{\mathcal{T}}(\tau)(\mathcal{T}) = (0_{\mathcal{H}})_{n\in\mathbb{Z}_+}$ the *minimal linear recurrence relation* of $\mathcal{T}$;
- $r := \deg(P_{\mathcal{T}})$ the *order* of the recursive sequence.

Now, let $x, y \in \mathcal{H}$, and consider the scalar sequence $\langle \mathcal{T}x, y\rangle := (\langle T_n x, y\rangle)_{n\in\mathbb{N}}$. This sequence is also a linear recursive sequence, and its minimal characteristic polynomial $P_{\langle \mathcal{T}x,y\rangle}$ satisfies

$$P_{\mathcal{T}} \text{ is a multiple of } P_{\langle \mathcal{T}x,y\rangle}. \tag{2.1}$$

Hence, we obtain

$$P_{\mathcal{T}}\mathbb{R}[X] = \bigcap_{x,y\in\mathcal{H}} P_{\langle \mathcal{T}x,y\rangle}\mathbb{R}[X]. \tag{2.2}$$

Remark 2.2. The set $\left\{P_{\langle \mathcal{T}x,y\rangle} \,:\, x, y \in \mathcal{H}\right\}$ is finite.

PROPOSITION 2.3. *With the above notation, we have:*

$$P_{\mathcal{T}} = \operatorname{lcm}\left\{P_{\langle \mathcal{T}x,y\rangle} \,:\, x, y \in \mathcal{H}\right\},$$

where lcm *denotes the least common multiple.*

PROPOSITION 2.4. *Let $\mathcal{Z}(P)$ denote the set of complex roots of a polynomial P. Then:*

1. $\mathcal{Z}(P_{\mathcal{T}}) = \bigcup_{x,y\in\mathcal{H}} \mathcal{Z}(P_{\langle \mathcal{T}x,y\rangle})$;
2. *$P_{\mathcal{T}}$ has only simple roots if and only if $P_{\langle \mathcal{T}x,y\rangle}$ has only simple roots, for every $x, y \in \mathcal{H}$.*

For more information, see [3].

Now we can state the Binet Formula for operator-valued sequences.

THEOREM 2.5. (OPERATOR-VALUED BINET FORMULA) *Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \subset \mathbf{B}(\mathcal{H})^{\mathbb{Z}_+}$ be a linear recursive sequence of bounded operators on a Hilbert space $\mathcal{H}$, and let $P \in \mathcal{P}(\mathcal{T})$ be a characteristic polynomial of $\mathcal{T}$.*

Suppose that P has distinct roots $\lambda_1, \ldots, \lambda_s$ with respective multiplicities $m_1, \ldots, m_s$. Then there exist unique bounded operators

$$\left\{S_{i,j} \in \mathbf{B}(\mathcal{H}) \,:\, 1 \le i \le s,\, 0 \le j \le m_i - 1\right\}$$

such that for all $n \ge 0$,

$$T_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} S_{i,j} n^j\right) \lambda_i^n.$$

Proof. Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+}$ be a linear recursive sequence of bounded operators, and let $P \in \mathcal{P}(\mathcal{T})$ be its characteristic polynomial. For any vectors $x, y \in \mathcal{H}$, the scalar sequence $(\langle T_n x, y\rangle)_{n\geq 0}$ also satisfies the linear recurrence relation determined by P.

By the classical scalar Binet formula (extended to multiple roots), there exist unique complex numbers $\beta_{i,j}(x,y)$ for $1 \leq i \leq s$ and $0 \leq j \leq m_i - 1$ such that

$$\langle T_n x, y\rangle = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j}(x,y) n^j \right) \lambda_i^n, \qquad \text{for all } n \geq 0.$$

For each fixed pair (i,j), the mapping

$$(x,y) \longmapsto \beta_{i,j}(x,y)$$

is a sesquilinear form on $\mathcal{H}\times\mathcal{H}$. Its continuity follows from the boundedness of the operators T_n and from the explicit dependence of the $\beta_{i,j}(x,y)$ on $\langle T_n x, y\rangle$ via an invertible Vandermonde-type system determined by the roots λ_i and their multiplicities. Hence, each $\beta_{i,j}$ is bounded.

By the Riesz representation theorem, there exists a unique bounded operator $S_{i,j} \in \mathbf{B}(\mathcal{H})$ such that

$$\beta_{i,j}(x,y) = \langle S_{i,j} x, y\rangle, \qquad \text{for all } x, y \in \mathcal{H}.$$

Substituting back, we get

$$\langle T_n x, y\rangle = \left\langle \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} S_{i,j} n^j \right) \lambda_i^n x, y \right\rangle,$$

and since this holds for all x, y, we conclude

$$T_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} S_{i,j} n^j \right) \lambda_i^n, \qquad \text{for all } n \geq 0.$$

∎

Remark 2.6. Let $(x_m)_m$ and $(y_m)_m$ be sequences in $\mathcal{H}$ such that $x_m \to x$ and $y_m \to y$ in $\mathcal{H}$. Then, by continuity of the operators T_n, we have

$$\langle T_n x, y\rangle = \lim_{m\to\infty} \langle T_n x_m, y_m\rangle.$$

Using the Binet formula for scalars on $\langle T_n x_m, y_m \rangle$, we get

$$\langle T_n x_m, y_m \rangle = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j}(x_m, y_m) n^j \right) \lambda_i^n.$$

Taking the limit as $m \to \infty$,

$$\langle T_n x, y \rangle = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \lim_{m\to\infty} \beta_{i,j}(x_m, y_m) n^j \right) \lambda_i^n.$$

By the uniqueness of the scalar Binet representation, it follows that

$$\lim_{m\to\infty} \beta_{i,j}(x_m, y_m) = \beta_{i,j}(x, y).$$

Hence, each sesquilinear form $\beta_{i,j}$ is continuous.

2.2. Operator-Valued Binet Formula and Representing Measure As a consequence of the Binet formula for a linear recurrence sequence (LRS) $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+}$, we establish in Theorem 2.7 the existence of a representing finitely atomic operator-valued charge as a solution for the moment problem associated to $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+}$. Additionally, we prove in Theorem 2.13 that this charge is an operator-valued measure if the associated sequence is positive type.

Theorem 2.7. *Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ be a linear recursive sequence (LRS) of self-adjoint operators. Then the following statements are equivalent:*

1. *The minimal polynomial $P_{\mathcal{T}}$ has only simple roots.*
2. *The sequence $\mathcal{T}$ admits a finitely atomic representing charge.*

In particular, if $\mathcal{Z}(P_{\mathcal{T}}) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, then there exist unique self-adjoint operators $S_1, S_2, \ldots, S_r \in \mathbf{B}_h(\mathcal{H})$ such that

$$T_n = \sum_{k=1}^{r} S_k \lambda_k^n, \qquad \text{for every } n \in \mathbb{N}.$$

That is, the finitely atomic representing charge has the form

$$E = \sum_{k=1}^{r} S_k \delta_{\lambda_k}.$$

Proof. (1) ⇒ (2) Assume that $P_{\mathcal{T}}(X) = \prod_{k=1}^{r}(X-\lambda_k)$. Then, according to the kernel lemma with $\tau^0 = \mathrm{id}$, we have:

$$\ker P_{\mathcal{T}}(\tau) = \ker(\tau-\lambda_1\mathrm{id}) \oplus \ker(\tau-\lambda_2\mathrm{id}) \oplus \cdots \oplus \ker(\tau-\lambda_r\mathrm{id}). \tag{2.3}$$

Since $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \ker P_{\mathcal{T}}(\tau)$, it follows from (2.3) that there exist unique sequences $\mathcal{T}^{(k)} = (T_n^{(k)})_{n\in\mathbb{Z}_+} \in \ker(\tau-\lambda_k\mathrm{id})$ for $k = 1,\dots,r$, such that:

$$\mathcal{T} = \mathcal{T}^{(1)} \oplus \cdots \oplus \mathcal{T}^{(r)} \iff T_n = T_n^{(1)} + \cdots + T_n^{(r)} \qquad \text{for all } n \in \mathbb{Z}_+.$$

Now, for each $k = 1,\dots,r$, we have:

$$\begin{aligned} \mathcal{T}^{(k)} \in \ker(\tau-\lambda_k\mathrm{id}) &\iff T_{n+1}^{(k)} = \lambda_k T_n^{(k)} \qquad \text{for all } n \in \mathbb{Z}_+, \\ &\iff T_n^{(k)} = \lambda_k^n T_0^{(k)} \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Define $S_k := T_0^{(k)} \in \mathbf{B}_h(\mathcal{H})$. Then:

$$T_n = \lambda_1^n S_1 + \lambda_2^n S_2 + \cdots + \lambda_r^n S_r, \qquad \text{for every } n \in \mathbb{Z}_+.$$

(2) ⇒ (1) Assume that there exist operators $S_1,\dots,S_r \in \mathbf{B}_h(\mathcal{H})$ such that

$$T_n = \lambda_1^n S_1 + \lambda_2^n S_2 + \cdots + \lambda_r^n S_r \qquad \text{for all } n \in \mathbb{Z}_+.$$

Let $P(X) := \prod_{k=1}^{r}(X-\lambda_k) = \sum_{k=0}^{r} a_k X^k$. Then:

$$\begin{aligned} P(\tau)(\mathcal{T}) &= \left(\sum_{k=0}^{r} a_k T_{n+k}\right)_{n\in\mathbb{N}} = \left(\sum_{k=0}^{r} a_k \sum_{j=1}^{r} \lambda_j^{n+k} S_j\right)_{n\in\mathbb{N}} \\ &= \left(\sum_{j=1}^{r}\left(\sum_{k=0}^{r} a_k\lambda_j^k\right)\lambda_j^n S_j\right)_{n\in\mathbb{N}} = \left(\sum_{j=1}^{r} P(\lambda_j)\lambda_j^n S_j\right)_{n\in\mathbb{N}} = (0_{\mathcal{H}})_{n\in\mathbb{N}}, \end{aligned}$$

since $P(\lambda_j) = 0$ for all $j = 1,\dots,r$. Thus, $P \in \mathcal{P}(\mathcal{T})$, and hence $P_{\mathcal{T}}$ divides P. Therefore, the minimal polynomial $P_{\mathcal{T}}$ has only simple roots. ∎

Remark 2.8. The implication (1) ⇒(2) is a direct corollary of Theorem 2.5.

In the sequel, let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ be an LRS with minimal polynomial $P_{\mathcal{T}}$ of degree r. For every non-zero $x \in \mathcal{H}$ and $n \in \mathbb{Z}_+$, we associate the local infinite and finite-type Hankel matrices, which are respectively defined as follows:

$$H(x) = \big(\langle T_{i+j}x, x\rangle_{\mathcal{H}}\big)_{i,j\in\mathbb{Z}_+} \quad \text{and} \quad H_n(x) = \big(\langle T_{i+j}x, x\rangle_{\mathcal{H}}\big)_{0\le i,j\le n}.$$

By identifying the polynomial $P(X) = \sum_{k=0}^{n} a_k X^k$ with the column vector $\widehat{P} =^t (a_0, \dots, a_n, 0, 0, \dots)$, we derive the following trivial lemma.

Lemma 2.9. *Using the above notation, we have $P \in \mathcal{P}(\mathcal{T}) \iff H(x)\widehat{P} = \widehat{0}$ for every $x \in \mathcal{H}$. In particular, $H(x)\widehat{P_{\mathcal{T}}} = \widehat{0}$ for every $x \in \mathcal{H}$.*

We also have the following conditions for positivity:

$$\begin{aligned} H(x) \ge 0 \quad &\iff \quad H_n(x) \ge 0 \qquad \text{for every } n \in \mathbb{Z}_+ \\ &\iff \quad {}^t\widehat{P} H(x)\widehat{P} \ge 0 \qquad \text{for every } P \in \mathbb{R}[X]. \end{aligned}$$

We derive the following crucial proposition.

Proposition 2.10. *Let $\mathcal{T}$ be a* LRS *of order r and $x \in \mathcal{H}$ be a non-zero vector. The following statements are equivalent.*

1. *$H(x) \ge 0$;*
2. *$H_{r-1}(x) \ge 0$.*

Proof. The direct implication is clear. To show the converse, let $x \neq 0$, be such that $H_{r-1}(x) \ge 0$. For every $P \in \mathbb{R}[X]$, we use the Euclidean division algorithm to write P as $P = QP_{\mathcal{T}} + R$ with $\deg(R) \le r-1$. It follows that

$${}^t\widehat{P} H(x)\widehat{P} =^t \widehat{R} H_{r-1}(x)\widehat{R} \ge 0.$$

Finally, $H(x) \ge 0$. ∎

Lemma 2.11. *Under the above notations, for all polynomials $A, B, C \in \mathbb{R}[X]$, the following statements hold:*

1. *For every non-zero $x \in \mathcal{H}$, we have ${}^t\widehat{A}\, H(x)\, \widehat{BC} = {}^t\widehat{AB}\, H(\infty)\, \widehat{C}$.*

2. *If $H(x) \geq 0$ (in the sense of positive semidefinite Hermitian operators), and there exists an integer $n \in \mathbb{N}$ such that $A^n \in \mathcal{P}(\mathcal{T})$, then $A \in \mathcal{P}(\mathcal{T})$.*

Proof. For $A, B, C \in \mathbb{R}[X]$, we write $A(X) = \sum_i a_i X^i$, $B(X) = \sum_j b_j X^j$, and $C(X) = \sum_k c_k X^k$. We have:

1. ${}^t\widehat{A} H(x) \widehat{BC} = \sum_{i,j,k} a_i b_j c_k \langle T_{i+j+k} x, x\rangle = {}^t\widehat{AB} H(x) \widehat{C}$,
2. If $H(x)\widehat{A^n} = \widehat{0} \Rightarrow H(x)\widehat{A} = \widehat{0}$. Indeed, the property is true for $n = 1$. For $n \geq 2$, using (1), we have ${}^t\widehat{A^{n-1}} H(x) \widehat{A^{n-1}} = {}^t\widehat{A^{n-2}} H(x) \widehat{A^n} = 0$. Then, the positivity of $H(x)$ gives us $H(x)\widehat{A^{n-1}} = \widehat{0}$. We conclude that $H(x)\widehat{A^n} = \widehat{0} \Rightarrow H(x)\widehat{A^{n-1}} = \widehat{0} \Rightarrow H(x)\widehat{A^{n-2}} = \widehat{0} \Rightarrow \cdots \Rightarrow H(x)\widehat{A} = \widehat{0}$. So from Lemma 2.9, $A^n \in \mathcal{P}(\mathcal{T})$, then $A \in \mathcal{P}(\mathcal{T})$. ∎

LEMMA 2.12. *Under the previous notations, if $H(x) \geq 0$ for every non-zero $x \in \mathcal{H}$, then the minimal polynomial $P_{\mathcal{T}}$ has simple (i.e., distinct) roots.*

Proof. Let $P_{\mathcal{T}}(X) = \prod_{i=1}^{s} (X - \lambda_i)^{m_i}$, and define $n = \max_{1\leq i\leq s} m_i$ (note that $s \leq r$).

Define $Q(X) = \prod_{i=1}^{s} (X - \lambda_i)$. Then,

$$Q^n(X) = \prod_{i=1}^{s} (X - \lambda_i)^n = \left(\prod_{i=1}^{s} (X - \lambda_i)^{n-m_i} \right) P_{\mathcal{T}}(X).$$

Hence, $Q^n \in \mathcal{P}_{\mathcal{T}}$. From Lemma 2.9, it follows that

$$H(x)\widehat{Q^n} = \widehat{0}.$$

By Lemma 2.11, we then deduce that $Q \in \mathcal{P}(\mathcal{T})$.

Since $P_{\mathcal{T}}$ is the minimal polynomial of $\mathcal{T}$, it must divide Q. But Q already divides $P_{\mathcal{T}}$, and both are monic polynomials. Therefore, $P_{\mathcal{T}} = Q$, and thus all the roots of $P_{\mathcal{T}}$ are simple. ∎

THEOREM 2.13. *Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ be a linear recurrence sequence (LRS) with minimal polynomial $P_{\mathcal{T}}$ of degree r. Then the following statements are equivalent:*

(1) *$\mathcal{T}$ is an operator moment sequence on $\mathcal{Z}(P_{\mathcal{T}})$;*

(2) *for every non-zero $x \in \mathcal{H}$, we have $H_{r-1}(x) \geq 0$;*

(3) *for every non-zero $x \in \mathcal{H}$, we have $\langle \mathcal{T}x, x\rangle$ is an scalar moment sequence on $\mathcal{Z}(P_{\mathcal{T}})$.*

More precisely, if

$$\mathcal{Z}(P_{\mathcal{T}}) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}.$$

Consequently, the representing operator-valued measure associated with $\mathcal{T}$ admits the expression

$$E = \sum_{k=1}^{r} S_k \delta_{\lambda_k}, \qquad \text{where } S_1, \dots, S_r \in \mathbf{B}(\mathcal{H})_+.$$

Proof. Clearly, (1) $\Rightarrow$ (2). It remains to show that (2) $\Rightarrow$ (1).

Suppose $H_{r-1}(x) \geq 0$ for every non-zero $x \in \mathcal{H}$. By Lemma 2.12, the set

$$\mathcal{Z}(P_{\mathcal{T}}) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$$

consists of simple roots.

According to Theorem 2.7, there exists a finitely atomic operator-valued charge

$$E = \sum_{k=1}^{r} S_k \delta_{\lambda_k}$$

such that

$$T_n = \lambda_1^n S_1 + \lambda_2^n S_2 + \cdots + \lambda_r^n S_r, \qquad \text{for all } n \in \mathbb{Z}_+.$$

For $i = 1, \dots, r$, we now prove that $S_i \geq 0$. Indeed, using the Lagrange interpolation polynomial

$$L_i(X) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{X - \lambda_j}{\lambda_i - \lambda_j},$$

we obtain

$${}^t\widehat{L_i} H_{r-1}(x) \widehat{L_i} = \langle S_i x, x\rangle_{\mathcal{H}} \geq 0$$

for every non-zero $x \in \mathcal{H}$. This shows that $S_i \in \mathbf{B}(\mathcal{H})_+$.

(2) $\iff$ (3) follows from Proposition 2.10 and the solvability of the scalar moment problem, which completes the proof. ∎

2.3. Two particular cases: Recursive sequences of order $r = 1$ or $r = 2$.

Example 2.14. $r = 1$: Let $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$, such that $T_{n+1} = \lambda T_n$ for every $n \in \mathbb{Z}_+$. Then, $\mathcal{T}$ is an operator moment sequence if and only if $T_0 \geq 0$.

In this case, the associated representing OVM is given by: $E = T_0\delta_\lambda$.

Using [19, Proposition 4.1] together with Theorem 2.13, we derive the following result.

Corollary 2.15. $r = 2$: *Let* $\mathcal{T} = (T_n)_{n\in\mathbb{Z}_+} \in \mathbf{B}_h(\mathcal{H})^{\mathbb{Z}_+}$ *be an* LRS *satisfying*

$$T_{n+2} = (\lambda_1 + \lambda_2)T_{n+1} - \lambda_1\lambda_2 T_n \qquad \text{with } \lambda_1 < \lambda_2.$$

Then the following statements are equivalent:

1. $\mathcal{T}$ *is an operator moment sequence;*
2. $\langle T_0x, x\rangle \geq 0$ *and* $\langle T_1x, x\rangle^2 - (\lambda_1 + \lambda_2)\langle T_1x, x\rangle + \lambda_1\lambda_2 \leq 0$;
3. *for every* $x \in \mathcal{H}$, $\begin{pmatrix} \langle T_0x, x\rangle & \langle T_1x, x\rangle \\ \langle T_1x, x\rangle & \langle T_2x, x\rangle \end{pmatrix} \geq 0$;
4. $T_0 \geq 0$, $T_2 \geq 0$ *and for every* $x \in \mathcal{H}$, *we have*
$$\langle T_1x, x\rangle^2 \leq \langle T_2x, x\rangle\langle T_0x, x\rangle.$$

Moreover, the associated representing OVM *is given by*

$$E = \frac{1}{\lambda_1 - \lambda_2}(T_1 - \lambda_2 T_0)\delta_{\lambda_1} + \frac{1}{\lambda_2 - \lambda_1}(T_1 - \lambda_1 T_0)\delta_{\lambda_2}.$$

Remark 2.16. To see that E is an OVM, it suffices to show that $\lambda_1 T_0 \leq T_1 \leq \lambda_2 T_0$. Indeed, from 3 in Corollary 2.15, we get for every $x \in \mathcal{H}$,

$$\begin{aligned}\big(\langle T_1x, x\rangle - \lambda_1\langle T_0x, x\rangle\big)\big(\langle T_1x, x\rangle - \lambda_2\langle T_0x, x\rangle\big)\\ = \langle T_1x, x\rangle^2 - \langle T_2x, x\rangle\langle T_0x, x\rangle \leq 0.\end{aligned}$$

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References

[1] S.K. Berberian, "Notes on spectral theory", Van Nostrand Mathematical Studies, No. 5, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[2] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, "Introduction to algorithms", Third edition, MIT Press, Cambridge, MA, 2009.

[3] R.E. Curto, A. Ech-charyfy, H. El Azhar, E.H. Zerouali, The local operator moment problem on $\mathbb{R}$, *Complex Anal. Oper. Theory* **19** (2) (2025), Paper No. 25, 31 pp.

[4] E.B. Davies, "Linear operators and their spectra",Cambridge Stud. Adv. Math., 106, Cambridge University Press, Cambridge, 2007.

[5] F. Dubeau, W. Motta, M. Rachidi, O. Saeki, On weighted r-generalized Fibonacci sequences, *Fibonacci Quart.* **35** (2) (1997), 102 – 110.

[6] K.-J. Engel, R. Nagel, "One-parameter semigroups for linear evolution equations", Grad. Texts in Math., 194, Springer-Verlag, New York, 2000.

[7] K. Idrissi, E.H. Zerouali, Charges solve the truncated complex moment problem, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **21** (4) (2018), no. 4, 1850027, 16 pp.

[8] K. Idrissi, E.H. Zerouali, Recursiveness approach to multi-dimensional moment problems, *Ann. Funct. Anal.* **13** (1) (2022), Paper No. 2, 19 pp.

[9] J.A. Jeske, Linear recurrence relations, Part I, *Fibonacci Quart.* **1** (2) (1963), 69 – 74.

[10] J.K. Hale, "Ordinary differential equations", Dover Publications, 2009.

[11] W.G. Kelley, A.C. Peterson, "Difference equations: An introduction with applications", Academic Press, Inc., Boston, MA, 1991.

[12] C. Levesque, On mth order linear recurrences, *Fibonacci Quart.* **23** (4) (1985) 290 – 293.

[13] M. Mlak, "Dilations of Hilbert space operators (general theory)", *Dissertationes Math. (Rozprawy Mat.)* **153** (1978), 61 pp.

[14] M. Mouline, M. Rachidi, Application of Markov chains properties to r-generalized Fibonacci sequences, *Fibonacci Quart.* **37** (1) (1999), 34 – 38.

[15] G.N. Philippou, On the kth order linear recurrence and some probability applications, in "Applications of Fibonacci numbers" (San Jose, CA, 1986), Kluwer Academic Publishers, Dordrecht, 1988, 89 – 96..

[16] P. Pietrzycki J. Stochel, Subnormal nth roots of quasinormal operators are quasinormal, *J. Funct. Anal.* **280** (12) (2021), Paper No. 109001, 14 pp.

[17] P. Pietrzycki J. Stochel, Two-moment characterization of spectral measures on the real line, *Canad. J. Math.* **75** (4) (2023), 1369 – 1392.

[18] K. Schmüdgen, "Unbounded Self-adjoint operators on Hilbert space", Grad. Texts in Math., 265, Springer, Dordrecht, 2012.

[19] R. Ben Taher, M. Rachidi, E.H. Zerouali, Recursive subnormal completion and the truncated moment problem, *Bull. London Math. Soc.* **33** (4) (2001), 425 – 432.

[20] W.J. Terrell, "Stability and stabilization. An introduction", Princeton University Press, Princeton, NJ, 2009.

[21] Y. Saad, "Iterative methods for sparse linear systems", Second edition, SIAM, Philadelphia, PA, 2003.