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Metrizability of the strong dual: equivalent topological characterizations

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Abstract: This short article presents several equivalent topological characterizations for the metrizability of the strong dual of a locally convex Hausdorff space. Among our key findings, we establish that the metrizability of the strong dual is precisely equivalent to it being a q-space.

Key words: Locally convex space, dual space, bounded sets, strong topology, metrizability.

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1. Introduction

Let X be a locally convex Hausdorff space (LCHS) over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), and let X^* denote the dual of X. In classical functional analysis, the space X^* is equipped with a number of important topologies. One of the most important topologies on the space X^* is the topology τ_s, of uniform convergence on bounded subsets of X (also known as the strong topology). When X is a normed space, this topology on X^* reduces to the classical dual norm topology. The space X^* equipped with the strong topology τ_s is called the strong dual of X. First, we recall the definition of the strong topology.

Definition 1.1. Let X be a locally convex Hausdorff space. Then the strong topology τ_s on X^* is generated by the family

$$\big\{\rho_B \,:\, B \text{ is a bounded subset of } X\big\}$$

of seminorms on X^*, where $\rho_B(f) = \sup\big\{|f(x)| \,:\, x \in B\big\}$ for each $f \in X^*$.

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We would like to mention that if $\mathcal{B}$ is a fundamental system of bounded sets in X, that is, for every bounded set A in X there exists $B \in \mathcal{B}$ such that $A \subseteq B$, then the family $\{\rho_B : B \in \mathcal{B}\}$ of seminorms defines the strong topology on X^*. Furthermore, the family $\{B^\circ : B \in \mathcal{B}\}$, where

$$B^\circ = \big\{ f \in X^* : \sup\{|f(x)| : x \in B\} \leq 1 \big\}$$

is the polar of B, forms a closed neighborhood base at the zero functional 0 in (X^*, τ_s). For more about locally convex spaces, we refer to [5] and [4].

It is well-known that the metrizability of a Hausdorff topological group is equivalent to the weaker notion of first countability. This article aims to show that many topological properties that are strictly weaker than that of the first countability, in an arbitrary topological space, are equivalent to the metrizability of the space (X^*, τ_s). Here, we would like to mention that since we are dealing with continuous linear maps, the classical Tietze's extension theorem, generally used for extending continuous (non-linear) functions, is no longer applicable. Instead we need to use the classical Hahn-Banach theorem. However, its application requires careful consideration as it necessitates the continuity of the linear map on a set that is not only closed but also convex. We first define various topological properties and discuss relations between them.

Definitions 1.2. A subset S of a space X is said to have *countable character* if there exists a countable collection $\{W_n : n \in \mathbb{N}\}$ of open sets in X such that $S \subseteq W_n$ for each n and if W is an open set containing S, then $W_n \subseteq W$ for some n. A space X is said to be of *(pointwise) countable type* if each (point) compact set is contained in a compact set having countable character.

A family of nonempty open sets in X is called a *π-base* if for every nonempty open set in X contains a member of this family. A space X is said to have a *countable π-character* if for each $x \in X$, there is a countable collection $\mathcal{B}_x$ of nonempty open sets in X such that each neighborhood of x contains some member of $\mathcal{B}_x$. Also $\mathcal{B}_x$ is called a *local π-base* at x. This is clearly weaker than first countability. Also it is easy to see that if a space X has a countable π-base, then it has a countable π-character. A space X is an *r-space* if each point of X has a sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods with the property that if $x_n \in V_n$ for each n, then the set $\{x_n : n \in \mathbb{N}\}$ is contained in a compact subset of X. A property weaker than being an r-space is that of being a q-space. A space X is called a *q-space* if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. Another property stronger than being a q-space is that of being an *M-space* ([2, p. 437]), which can be characterized as a space that can be mapped onto a metric space by a *quasi-perfect map* (a continuous closed map in which inverse images of points are countably compact).

A first countable space is of pointwise countable type and every pointwise countable type space is an r-space. Also every metrizable space is of countable type as well as an M-space. For more details on these properties see [2], [3] and [6].

As is customary to locally convex spaces while considering topological properties which are defined on the points, we simply need to define them at 0. In particular, for an LCHS, properties such as countable π-character, being an r-space, and a q-space may be defined only at 0.

2. Main result

In order to relate the metrizability of (X^*, τ_s) with the topological properties discussed above, we need the following results.

LEMMA 2.1. *Let D be a dense subset of a Tychonoff (completely regular and Hausdorff) space X and $x \in D$. Then x has a countable local π-base in D if and only if x has a countable local π-base in X.*

Proof. Suppose x has a countable local π-base in D. So there exists a countable collection $\{U_n : n \in \mathbb{N}\}$ of nonempty open sets in D such that any neighborhood of x in D contains some U_n. For each n, there exists a non-empty open set W_n in X such that $U_n = W_n \cap D$. Now let W be any neighborhood of x in X. Then there exists an open neighborhood V of x in X such that $V \subseteq \overline{V} \subseteq W$. Since $V \cap D$ is a neighborhood of x in D, there exists U_n such that $U_n \subseteq V \cap D$. Since D is dense in X and W_n is open in X, $\overline{U_n} = \overline{W_n \cap D} = \overline{W_n}$. So $W_n \subseteq \overline{W_n} = \overline{U_n} \subseteq \overline{V \cap D} \subseteq \overline{V} \subseteq W$. The proof of the converse is trivial. ∎

The following lemma can be proved in a manner similar to Lemma 2.1.

LEMMA 2.2. *Let D be a dense subset of a Tychonoff space X, and let A be a compact subset of D. Then A has countable character in D if and only if A has countable character in X.*

The following result is well-known.

Lemma 2.3. *The following statements are true in a locally convex space.*

1. *The closure of a bounded set is bounded.*
2. *The convex hull of a bounded set is bounded.*
3. *The balanced hull of a bounded set is bounded.*

A sequence $\{B_n : n \in \mathbb{N}\}$ of bounded sets in an LCHS X is called a *fundamental sequence of bounded sets* if $\{B_n : n \in \mathbb{N}\}$ is a fundamental system of bounded sets.

Theorem 2.4. *For a locally convex Hausdorff space X, the following conditions are equivalent:*

(a) (X^*, τ_s) *is metrizable;*

(b) (X^*, τ_s) *is first countable;*

(c) (X^*, τ_s) *has a countable π-character;*

(d) (X^*, τ_s) *contains a dense subspace which has a countable π-character;*

(e) (X^*, τ_s) *is of countable type;*

(f) (X^*, τ_s) *is pointwise countable type;*

(g) (X^*, τ_s) *has a dense subspace of pointwise countable type;*

(h) (X^*, τ_s) *is an M-space;*

(i) (X^*, τ_s) *is r-space;*

(j) (X^*, τ_s) *is q-space;*

(k) X *has a fundamental sequence of bounded sets.*

Proof. From the earlier discussions, we have (a) $\Rightarrow$ (h) $\Rightarrow$ (j), (i) $\Rightarrow$ (j), (a) $\Rightarrow$ (e), and (f) $\Rightarrow$ (i). Also (a) $\iff$ (b) is true in any locally convex Hausdorff space, and (c) $\Rightarrow$ (d), (e) $\Rightarrow$ (f) $\Rightarrow$ (g) are immediate. By [1, Proposition 5.2.6, p. 298] (b) $\iff$ (c).

(d) $\Rightarrow$ (c): Let D be a dense subset of (X^*, τ_s) which has a countable π-character. Let $y \in D$ be arbitrary. Since y has a countable π-base in D, by Lemma 2.1 y has a countable π-base in (X^*, τ_s) also. So there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open sets in (X^*, τ_s) such that whenever U is an open set containing y, then $U_n \subseteq U$ for some n. Let $x \in (X^*, \tau_s)$ be arbitrary. Since (X^*, τ_s) is locally convex, there exists a homeomorphism $f : (X^*, \tau_s) \to (X^*, \tau_s)$ such that $f(y) = x$. Then $\{f(U_n) : n \in \mathbb{N}\}$ is a sequence of open sets in (X^*, τ_s). Let V be an open set containing x. Then $y \in f^{-1}(V)$. Consequently, there exists an n such that $U_n \subseteq f^{-1}(V)$, that is, $f(U_n) \subseteq V$. Hence $\{f(U_n) : n \in \mathbb{N}\}$ is a countable π-base at x in (X^*, τ_s).

(g) ⇒ (f): It can be proved in a manner similar to (d) ⇒ (c). But here instead of Lemma 2.1, we need to use Lemma 2.2.

(j) ⇒ (k): Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of neighborhoods of zero functional in the strong topology that satisfies the definition of a q-space. For each $n \in \mathbb{N}$, there exists a bounded set B_n in X such that $B_n^\circ \subseteq U_n$. By Lemma 2.3, we can assume that each B_n is closed, convex and balanced. Then $\{B_1, 2B_2, \ldots, nB_n, \ldots\}$ is a family of closed, bounded, convex, and balanced sets in X. Let B be any other bounded set in X. If possible, suppose $B \nsubseteq nB_n$ for any $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, we can choose $x_n \in B \setminus nB_n$. Then for each $n \in \mathbb{N}$, we can find $g_n \in X^*$ such that $|g_n(x)| \leq 1$ for all $x \in nB_n$ and $g_n(x_n) > 1$ (see, [5, Theorem 3.7]). Let $f_n = ng_n$. Then $f_n \in X^*$ for all n. Note that for each $n \in \mathbb{N}$, we have $|f_n(x)| \leq 1$ for all $x \in B_n$ and $f_n(x_n) > n$. So $f_n \in U_n$. If (f_n) has a cluster point f in (X^*, τ_s), then for each $k \in \mathbb{N}$, we can find $n_k > k$ such that $f_{n_k} \in f + B^\circ$. Consequently, for each k, we have $f(x_{n_k}) \geq k$. Which contradicts that f is continuous.

(k) ⇒ (a): Suppose $\{B_n : n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets in C. Then it is easy to see that the countable collection $\{\rho_{B_n} : n \in \mathbb{N}\}$ of seminorms generates the topology τ_s. Hence (X^*, τ_s) is metrizable. ∎

Remark 2.5. The above result relates various topological properties, as given in Definitions 1.2, of the space (X^*, τ_s) with the functional analytical property of existence of fundamental sequence of bounded sets in the locally convex space X.

We now give some examples of non-metrizable q-spaces.

EXAMPLE 2.6. Consider the space $\mathbb{R}$ equipped with the discrete topology. Let $\mathbb{R}_1$ denote its one-point compactification. Then the space $\mathbb{R}_1$ is a compact Hausdorff space which is a q-space but not first countable (see [6, Section 4, Example 1, p. 36]). Consequently, $\mathbb{R}_1$ is a non-metrizable q-space.

EXAMPLE 2.7. Let $X = I^I$ be the uncountable product of the closed unit interval $I = [0, 1] : I^I = \prod_{i \in I} I_i$ with the product topology. Then X is a q-space which is not first countable (see [6, Section 4, Example 53, p. 40] and [7, Example 105]).

More generally, every non-metrizable compact space will work as an example of a space which is a q-space but not metrizable.

Recall that every regular q-space in which each singleton set is a G_δ-set (countable intersection of open sets) is first countable (see [3, Theorem 1.7.7, p. 38]). So it is interesting to know that whether there exists a locally convex Hausdorff space X which is a q-space but in which singleton sets are not G_δ. Note that in a locally convex Hausdorff space X each singleton set is G_δ if and only if $\{0\}$ is G_δ. For the space (X^*, τ_s), we have the following interesting result.

THEOREM 2.8. *For a locally convex Hausdorff space X, the following conditions are equivalent:*

(a) *every singleton set in (X^*, τ_s) is G_δ;*

(b) *there exists a sequence $\{B_n : n \in \mathbb{N}\}$ of closed, bounded, convex and balanced sets in X such that $B_n \subseteq B_{n+1}$ and $X = \overline{\cup_{n=1}^{\infty} B_n}$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\{0\}$ is a G_δ-set in (X^*, τ_s). So there exists a sequence (F_n) of closed, bounded, convex and balanced sets in X such that $\{0\} = \cap_{n=1}^{\infty} F_n^\circ$. Now define $B_1 = F_1$, and for $n \geq 2$, let $B_n =$ Closed balanced and convex hull of $\cup_{i=1}^{n} F_i$. Then note that each B_n is bounded and $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Also $F_n \subseteq B_n$ implies $B_n^\circ \subseteq F_n^\circ$ for all $n \in \mathbb{N}$. So $\{0\} = \cap_{n=1}^{\infty} B_n^\circ = \cap_{n=1}^{\infty} F_n^\circ$. We show that $X = \overline{\cup_{n=1}^{\infty} B_n}$. Suppose $x \in X \setminus \overline{\cup_{n=1}^{\infty} B_n}$. Since $\overline{\cup_{n=1}^{\infty} B_n}$ is a closed convex set, by Hahn-Banach theorem there exists $f \in X^*$ such that $f(x) > 0$ and $f(y) = 0$ for all $y \in \overline{\cup_{n=1}^{\infty} B_n}$. Then $f \in B_n^\circ$ for each $n \in \mathbb{N}$. Hence $f \in \cap_{n=1}^{\infty} B_n^\circ$, and thus $f = 0$. We arrive at a contradiction.

(b) $\Rightarrow$ (a): Suppose there exists a countable family $\{B_n : n \in \mathbb{N}\}$ satisfying condition (b). Then it is easy to see that $\{0\} = \cap_{n=1}^{\infty} B_n^\circ$. Hence $\{0\}$ is G_δ in (X^*, τ_s). ∎

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