



Four-dimensional quasi-Einstein non-reductive homogeneous spaces are Einstein

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Abstract: We investigate quasi-Einstein structures on four-dimensional non-reductive homogeneous spaces. We show that contrary to the Ricci solitons structures, quasi-Einstein structures display a strong rigidity in the sense that every such a structure is necessarily Einstein.

Key words: non-reductive, homogeneous space, quasi-Einstein.

MSC (2020): 53C25, 53B20, 53C30.

1. INTRODUCTION AND MAIN RESULT

A pseudo-Riemannian homogeneous manifold (M, g) is said to be reductive if it can be realized as a coset space $M = G/H$, such that the Lie algebra \mathfrak{g} can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where \mathfrak{m} is an $Ad(H)$ -invariant subspace of \mathfrak{g} , and it is said to be non-reductive if it does not admit such a decomposition. It is well known that Riemannian homogeneous spaces are reductive and two- and three-dimensional pseudo-Riemannian homogeneous spaces are reductive. While for the four-dimensional pseudo-Riemannian case, there exist non-reductive homogeneous spaces which were classified in [8]. The geometry of these spaces has been investigated in many papers showing very interesting results (cf. [1], [3], [2], [4]).

On the other hand, a Lorentzian manifold (M, g) is said to be a quasi-Einstein manifold if there exists a constant $\lambda \in \mathbb{R}$ and a vector field $X \in \mathfrak{X}(M)$ such that

$$Ric(g) + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^b \otimes X^b = \lambda g,$$

where m is a non-zero real constant, Ric is the Ricci tensor of g , \mathcal{L}_X denotes the Lie derivative with respect to X and X^b the one-form dual to X with respect to g .

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Quasi-Einstein manifolds reduce to

- Einstein manifolds, when $X = 0$,
- Ricci solitons for $m = \infty$,
- gradient quasi-Einstein manifolds when $X = \nabla f$ is a gradient vector field, for some smooth function $f \in C^\infty(M)$, in that case the quasi-Einstein equation becomes $Ric_f^m = \lambda g$, where $Ric_f^m := Ric + Hess(f) - \frac{1}{m} df \otimes df$ is the m -Bakry-Émery Ricci tensor.

This last case is of particular interest, since the Hawking–Penrose singularity theorem and the timelike splitting theorem, which hold under some energy conditions, can be generalized to some positivity assumptions on the Bakry–Emery Ricci tensor (cf. [6, 9]).

Quasi-Einstein equations also characterize Levi-Civita connections which are projectively equivalent to affine connections with skew-symmetric Ricci tensor, for $m = 1 - n$ and $\lambda = 0$, where $n = \dim M$.

In [4], the authors provided a complete classification of Ricci solitons and Yamabe solitons on four-dimensional non-reductive homogeneous spaces. In [1], a classification was obtained for homogeneous generalized Ricci solitons, namely those generalized Ricci solitons for which the vector field X is invariant.

In the present work, we extend this line of investigation by classifying quasi-Einstein structures on four-dimensional non-reductive homogeneous pseudo-Riemannian spaces. Our main result can be stated as follows:

MAIN THEOREM 1.1. *Let (M, g) be a 4-dimensional non-reductive pseudo-Riemannian homogeneous space. Then (M, g) is a quasi-Einstein manifold if, and only if, (M, g) is an Einstein manifold.*

2. 4-DIMENSIONAL NON-REDUCTIVE HOMOGENEOUS SPACES

We recall that four-dimensional non-reductive pseudo-Riemannian spaces are described by the Table 1 on page 255 (cf. [4]).

In local coordinates, 4-dimensional non-reductive homogeneous pseudo-Riemannian spaces are described as follows:

Non-reductive pair $(\mathfrak{g}, \mathfrak{h})$	Commutation relations	\mathfrak{h} -invariant metrics																																																																
<p>A1 :</p> <p>$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$</p> <p>$\mathfrak{h} \cong \mathbb{R}$</p>	<table border="1"> <thead> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> <th>e_4</th> <th>e_5</th> </tr> </thead> <tbody> <tr> <td>e_1</td> <td>0</td> <td>$2e_2$</td> <td>$-2e_3$</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_2</td> <td>.</td> <td>0</td> <td>e_1</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_3</td> <td>.</td> <td>.</td> <td>0</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_4</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> <td>e_4</td> </tr> <tr> <td>e_5</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> </tr> </tbody> </table> <p>$\mathfrak{h} = \text{Span}(e_3 + e_4)$, $\mathfrak{g} = \text{Span}(e_1, e_2, e_5, e_3 - e_4)$</p>		e_1	e_2	e_3	e_4	e_5	e_1	0	$2e_2$	$-2e_3$	0	0	e_2	.	0	e_1	0	0	e_3	.	.	0	0	0	e_4	.	.	.	0	e_4	e_5	0	$\begin{pmatrix} a & 0 & -\frac{a}{2} & 0 \\ 0 & b & c & a \\ -\frac{a}{2} & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}$ <p>Lorentzian if $a(a - 4d) < 0$ neutral if $a(a - 4d) > 0$</p>																												
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<p>A2 :</p> <p>$\mathfrak{g} = A_{5,30}, \alpha \in \mathbb{R}$</p> <p>$\mathfrak{h} \cong \mathbb{R}$</p>	<table border="1"> <thead> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> <th>e_4</th> <th>e_5</th> </tr> </thead> <tbody> <tr> <td>e_1</td> <td>0</td> <td>0</td> <td>0</td> <td>0</td> <td>$(\alpha + 1)e_1$</td> </tr> <tr> <td>e_2</td> <td>.</td> <td>0</td> <td>0</td> <td>e_1</td> <td>αe_2</td> </tr> <tr> <td>e_3</td> <td>.</td> <td>.</td> <td>0</td> <td>e_2</td> <td>$(\alpha - 1)e_3$</td> </tr> <tr> <td>e_4</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> <td>e_4</td> </tr> <tr> <td>e_5</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> </tr> </tbody> </table> <p>$\mathfrak{h} = \text{Span}(e_4)$, $\mathfrak{g} = \text{Span}(e_1, e_2, e_3, e_5)$</p>		e_1	e_2	e_3	e_4	e_5	e_1	0	0	0	0	$(\alpha + 1)e_1$	e_2	.	0	0	e_1	αe_2	e_3	.	.	0	e_2	$(\alpha - 1)e_3$	e_4	.	.	.	0	e_4	e_5	0	$\begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}$ <p>Lorentzian if $ad > 0$ neutral if $ad < 0$</p>																												
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<p>A3 :</p> <p>$\mathfrak{g} = \begin{cases} A_{5,37}, & \text{if } \varepsilon = 1 \\ A_{5,36}, & \text{if } \varepsilon = -1 \end{cases}$</p> <p>$\mathfrak{h} \cong \mathbb{R}$</p>	<table border="1"> <thead> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> <th>e_4</th> <th>e_5</th> </tr> </thead> <tbody> <tr> <td>e_1</td> <td>0</td> <td>0</td> <td>0</td> <td>$2e_1$</td> <td>0</td> </tr> <tr> <td>e_2</td> <td>.</td> <td>0</td> <td>e_1</td> <td>e_2</td> <td>$-\varepsilon e_3$</td> </tr> <tr> <td>e_3</td> <td>.</td> <td>.</td> <td>0</td> <td>e_3</td> <td>e_2</td> </tr> <tr> <td>e_4</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_5</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> </tr> </tbody> </table> <p>$\mathfrak{h} = \text{Span}(e_3)$, $\mathfrak{g} = \text{Span}(e_1, e_2, e_4, e_5)$</p>		e_1	e_2	e_3	e_4	e_5	e_1	0	0	0	$2e_1$	0	e_2	.	0	e_1	e_2	$-\varepsilon e_3$	e_3	.	.	0	e_3	e_2	e_4	.	.	.	0	0	e_5	0	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix}$ <p>Lorentzian if $ab > 0$ neutral if $ab < 0$</p>																												
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<p>B3 :</p> <p>$\mathfrak{g} = (\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}^2) \times \mathbb{R}$,</p> <p>$\mathfrak{h} \cong \mathbb{R}^2$</p>	<table border="1"> <thead> <tr> <th></th> <th>e_1</th> <th>e_2</th> <th>e_3</th> <th>e_4</th> <th>e_5</th> <th>e_6</th> </tr> </thead> <tbody> <tr> <td>e_1</td> <td>0</td> <td>$-e_1$</td> <td>e_4</td> <td>0</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_2</td> <td>.</td> <td>0</td> <td>$-2e_3$</td> <td>$-e_4$</td> <td>$-e_1$</td> <td>$2e_6$</td> </tr> <tr> <td>e_3</td> <td>.</td> <td>.</td> <td>0</td> <td>0</td> <td>e_4</td> <td>e_2</td> </tr> <tr> <td>e_4</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> <td>0</td> <td>$-e_1$</td> </tr> <tr> <td>e_5</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> <td>0</td> </tr> <tr> <td>e_6</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>.</td> <td>0</td> </tr> </tbody> </table> <p>$\mathfrak{h} = \text{Span}(e_5, e_6)$, $\mathfrak{g} = \text{Span}(e_1, e_2, e_3, e_4)$</p>		e_1	e_2	e_3	e_4	e_5	e_6	e_1	0	$-e_1$	e_4	0	0	0	e_2	.	0	$-2e_3$	$-e_4$	$-e_1$	$2e_6$	e_3	.	.	0	0	e_4	e_2	e_4	.	.	.	0	0	$-e_1$	e_5	0	0	e_6	0	$\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}$ <p>neutral, $a \neq 0$</p>															
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Table 1: 4-dimensional non-reductive homogeneous pseudo-Riemannian spaces

THEOREM 2.1. ([3]) *Let M be a simply connected non-reductive pseudo-Riemannian homogeneous four-manifold. If M is not of type A5, then (M, g) is isometric to $(\mathbb{R}^4(x_1, x_2, x_3, x_4), g)$ where g is described in term of some real constants a, b, c and d as follows:*

$$\begin{aligned} A1: \quad g = & (4bx_2^2 + a)dx_1^2 + 4bx_2dx_1dx_2 - (4ax_2x_4 - 4cx_2 + a)dx_1dx_3 \\ & + 4ax_2dx_1dx_4 + bdx_2^2 - 2(ax_4 - c)dx_2dx_3 + 2adx_2dx_4 + d.dx_3^2, \end{aligned}$$

on the whole of \mathbb{R}^4 , whenever $a(a - 4d) \neq 0$.

$$\begin{aligned} A2: \quad g = & -2ae^{2\alpha x_4}dx_1dx_3 + ae^{2\alpha x_4}dx_2^2 + be^{2(\alpha-1)x_4}dx_3^2 \\ & + 2ce^{(\alpha-1)x_4}dx_3dx_4 + d.dx_4^2, \end{aligned}$$

on the whole of \mathbb{R}^4 , whenever $ad \neq 0$.

$$A3: \quad g = 2ae^{2x_3}dx_1dx_4 + ae^{2x_3}\cos(x_4)^2dx_2^2 + bdx_3^2 + 2cdx_3dx_4 + d.dx_4^2,$$

if $\varepsilon = 1$, on the open subset where $\cos(x_4) \neq 0$, whenever $ab \neq 0$,

$$A3: \quad g = 2ae^{2x_3}dx_1dx_4 + ae^{2x_3}\cosh(x_4)^2dx_2^2 + bdx_3^2 + 2cdx_3dx_4 + d.dx_4^2,$$

if $\varepsilon = -1$, on the whole \mathbb{R}^4 , whenever $ab \neq 0$.

$$\begin{aligned} A4: \quad g = & \left(\frac{a}{2}x_4^2 + 4bx_2^2 + a\right)dx_1^2 + 4bx_2dx_1dx_2 + ax_2(4 + x_4^2)dx_1dx_3 \\ & + a(1 + 2x_2x_3)x_4dx_1dx_4 + bdx_2^2 \\ & + \frac{a}{2}(4 + x_4^2)dx_2dx_3 + ax_3x_4dx_2dx_4 + \frac{a}{2}dx_4^2, \end{aligned}$$

on the whole \mathbb{R}^4 , whenever $a \neq 0$.

$$\begin{aligned} B1: \quad g = & (d(x_3^2 + 4x_2x_3x_4 + 4x_2^2x_4^2) + 4cx_2x_3 + 8cx_2^2x_4 + 2ax_3 + 4bx_2^2)dx_1^2 \\ & + 2(d(x_3x_4 + 2x_2x_4^2) + 4cx_2x_4 + cx_3 + 2bx_2)dx_1dx_2 \\ & + 2(d(x_3 + 2x_2x_4) + 2cx_2 + a)dx_1dx_3 + 4ax_2dx_1dx_4 \\ & + (d.x_4^2 + 2cx_4 + b)dx_2^2 + 2(d.x_4 + c)dx_2dx_3 + 2adx_2dx_4 + d.dx_3^2, \end{aligned}$$

on the whole \mathbb{R}^4 , whenever $a \neq 0$.

$$\begin{aligned} B2: \quad g = & \left(a - \frac{a}{2}x_4^2 + 4bx_2^2\right)dx_1^2 + 4bx_2dx_1dx_2 - ax_2(x_4^2 - 4)dx_1dx_3 \\ & - a(1 + 2x_2x_3)x_4dx_1dx_4 + bdx_2^2 \\ & - \frac{a}{2}(x_4^2 - 4)dx_2dx_3 - ax_3x_4dx_2dx_4 - \frac{a}{2}dx_4^2, \end{aligned}$$

on the open dense subset of \mathbb{R}^2 where $x_4 \neq \pm 2$, whenever $a \neq 0$.

$$B3: \quad g = -2ae^{-x_2}x_3dx_1dx_2 + 2ae^{-x_2}x_3dx_1dx_3 + 2(2bx_3^2 - ax_4)dx_2^2 \\ - 4bx_3dx_2dx_3 + 2adx_2dx_4 + bdx_3^2,$$

on the whole of \mathbb{R}^4 , whenever $a \neq 0$. If M is of type A5, then there exists a real constant $a \neq 0$, such that M is locally isometric to $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}^2$, and the invariant metric on A5 takes the form

$$A5: \quad g = -\frac{ax_4}{4x_2}dx_1dx_2 + \frac{a}{4}dx_1dx_4 + \frac{a(2 + 2x_1x_4 + x_3^2)}{8x_2^2}dx_2^2 \\ - \frac{ax_3}{4x_2}dx_2dx_3 - \frac{ax_1}{4x_2}dx_2dx_4 + \frac{a}{8}dx_3^2,$$

on the open subset where $x_2 \neq 0$.

Remark 2.2. For A1, A2 and A3 the metric g has either a neutral or a Lorentzian signature, for A4 and A5 it has a Lorentzian signature, while for B1, B2 and B3 it is always neutral.

We deduce, from Main Theorem 1.1 and [2], the following characterization of 4-dimensional quasi-Einstein non-reductive homogeneous spaces (cf. also [4], [5]):

PROPOSITION 2.3. *An invariant metric g of a non-reductive homogeneous four-manifold $M = G/H$ is a quasi-Einstein metric if, and only if, one of the following conditions hold*

- (i) M is of type A2 and either $\alpha = 2/3$ or g satisfies $b = 0$;
- (ii) M is of type A3 and g satisfies $d = -\varepsilon b$;
- (iii) M is of type A4 and g satisfies $b = 0$;
- (iv) M is of type A5;
- (v) M is of type B1 and g satisfies $bd = c^2$;
- (vi) M is of type B2 and g satisfies $b = 0$;
- (vii) M is of type B3.

3. PROOF OF MAIN THEOREM

Let g be an invariant pseudo-Riemannian metric of a non-reductive homogeneous pseudo-Riemannian manifold, $X = \sum_i X_i \partial_i \in \mathfrak{X}(M)$ be a vector field over M and let $(\mathcal{L}_X g)_{ij}$ denote $(\mathcal{L}_X g)_{ij} := (\mathcal{L}_X g)(\partial_i, \partial_j)$. We shall discuss the eight classes from Table 1:

A1: The Lie derivative of g with respect to X is described by

$$\left\{ \begin{array}{l} (\mathcal{L}_X g)_{11} = 2[(4bx_2^2 + a)\partial_1 X_1 + 2bx_2(\partial_1 X_2 + 2X_2) + 2ax_2\partial_1 X_4 \\ \quad - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_1 X_3], \\ (\mathcal{L}_X g)_{12} = 2bx_2\partial_1 X_1 + b\partial_1 X_2 - (ax_4 - c)\partial_1 X_3 + a\partial_1 X_4 + (4bx_2^2 + a)\partial_2 X_1 \\ \quad + 2bx_2\partial_2 X_2 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_2 X_3 + 2ax_2\partial_2 X_4 + 2bX_2, \\ (\mathcal{L}_X g)_{13} = (2ax_2x_4 - 2cx_2 + \frac{a}{2})(\partial_1 X_1 - \partial_3 X_3) - (ax_4 - c)(\partial_1 X_2 + 2X_2) \\ \quad + d\partial_1 X_3 + (4bx_2^2 + a)\partial_3 X_1 + 2bx_2\partial_3 X_2 + 2ax_2(\partial_3 X_4 - X_4), \\ (\mathcal{L}_X g)_{14} = 2ax_2(\partial_1 X_1 + \partial_4 X_4) + a\partial_1 X_2 + (4bx_2^2 + a)\partial_4 X_1 + 2bx_2\partial_4 X_2 \\ \quad - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_4 X_3 + 2aX_2, \\ (\mathcal{L}_X g)_{22} = 2[2bx_2\partial_2 X_1 + b\partial_2 X_2 - (ax_4 - c)\partial_2 X_3 + a\partial_2 X_4], \\ (\mathcal{L}_X g)_{23} = -(2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_2 X_1 - aX_4 + 2bx_2\partial_3 X_1 \\ \quad - (ax_4 - c)(\partial_2 X_2 + \partial_3 X_3) + b\partial_3 X_2 + d\partial_2 X_3 + a\partial_3 X_4, \\ (\mathcal{L}_X g)_{24} = 2ax_2\partial_2 X_1 + a\partial_2 X_2 + \partial_4 X_4 + 2bx_2\partial_4 X_1 \\ \quad + b\partial_4 X_2 - (ax_4 - c)\partial_4 X_3, \\ (\mathcal{L}_X g)_{33} = -(4ax_2x_4 - 4cx_2 + a)\partial_3 X_1 + 2(ax_4 - c)\partial_3 X_2 + 2d\partial_3 X_3, \\ (\mathcal{L}_X g)_{34} = 2ax_2\partial_3 X_1 + a\partial_3 X_2 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_4 X_1 \\ \quad - (ax_4 - c)\partial_4 X_2 + d\partial_4 X_3, \\ (\mathcal{L}_X g)_{44} = 4ax_2\partial_4 X_1 + 2a\partial_4 X_2. \end{array} \right.$$

The Ricci tensor of A1 is described by (cf. [4])

$$Ric = \begin{pmatrix} \frac{8b(a+12d)}{a(a-4d)}x_2^2 - 2 & \frac{4b(a+12d)}{a(a-4d)}x_2 & \frac{1}{a}(4ax_2x_4 - 4cx_2 + a) & -4x_2 \\ \frac{4b(a+12d)}{a(a-4d)}x_2 & \frac{2b(a+12d)}{a(a-4d)} & \frac{2}{a}(ax_4 - c) & -2 \\ \frac{1}{a}(4ax_2x_4 - 4cx_2 + a) & \frac{2}{a}(ax_4 - c) & -\frac{1}{2} & 0 \\ -4x_2 & -2 & 0 & 0 \end{pmatrix}.$$

Hence the quasi-Einstein equations for (M, g, X) are given as follows:

$$\left. \begin{aligned}
 &2m(4abx_2^2 + 48dbx_2^2 - a^2 + 4ad) - m\lambda a(a - 4d)(4bx_2^2 + a) \\
 &+ am(a - 4d)[2ax_2\partial_1 X_4 + (4bx_2^2 + a)\partial_1 X_1 + 2bx_2(\partial_1 X_2 + 2X_2) \\
 &- (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_1 X_3] - a(a - 4d)[(4bx_2^2 + a)X_1 + 2bx_2X_2 \\
 &- (4ax_2x_4 - 2cx_2 + \frac{a}{2})X_3 + 2ax_2X_4]^2 = 0, \\
 &4bm x_2[2(a + 12d) - \lambda a(a - 4d)] + am(a - 4d)[2bx_2\partial_1 X_1 + b\partial_1 X_2 \\
 &- (ax_4 - c)\partial_1 X_3 + a\partial_1 X_4 + (4bx_2^2 + a)\partial_2 X_1 + 2x_2(b\partial_2 X_2 + a\partial_2 X_4) \\
 &- (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_2 X_3 + 2bX_2] - a(a - 4d)[2bx_2X_1 + bX_2 \\
 &- (ax_4 - c)X_3 + aX_4][4x_2(aX_4 + bX_2) \\
 &- (4ax_2x_4 - 4cx_2 + a)X_3 + 2(4bx_2^2 + a)X_1] = 0, \\
 &m(2 + a\lambda)(4ax_2x_4 - 4cx_2 + a) + am[(2ax_2x_4 - 2cx_2 + \frac{a}{2})(\partial_1 X_1 - \partial_3 X_3) \\
 &+ d\partial_1 X_3 - (ax_4 - c)(\partial_1 X_2 + 2X_2) + (4bx_2^2 + a)\partial_3 X_1 + 2ax_2(\partial_3 X_4 - X_4) \\
 &+ 2bx_2\partial_3 X_2] + 2a[(4bx_2^2 + a)X_1 + 2x_2(bX_2 + aX_4) - (2ax_2x_4 \\
 &- 2cx_2 + \frac{a}{2})X_3][(ax_4 - c)X_2 + (2ax_2x_4 - 2cx_2 + \frac{a}{2})X_1 - dX_3] = 0 \\
 &4m x_2(2 + a\lambda) - m[2ax_2(\partial_1 X_1 + \partial_4 X_4) + a\partial_1 X_2 + (4bx_2^2 + a)\partial_4 X_1 \\
 &+ 2bx_2\partial_4 X_2] + 2a[(4bx_2^2 + a)X_1 + 2bx_2X_2 \\
 &- (2ax_2x_4 - 2cx_2 + \frac{a}{2})X_3 + 2ax_2X_4][2x_2X_1 + X_2] = 0, \\
 &2mb(a + 12d) + ma(a - 4d)[2bx_2\partial_2 X_1 + b\partial_2 X_2 - (ax_4 - c)\partial_2 X_3 + a\partial_2 X_4] \\
 &- a(a - 4d)[2bx_2X_1 + bX_2 - (ax_4 - c)X_3 + aX_4]^2 - am(a - 4d)\lambda b = 0, \\
 &2m(2 + a\lambda)(ax_4 - c) + am[(2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_2 X_1 + a(X_4 - \partial_3 X_4) \\
 &- 2bx_2\partial_3 X_1 + (ax_4 - c)(\partial_2 X_2 + \partial_3 X_3) - b\partial_3 X_2 - d\partial_2 X_3] + a[2bx_2X_1 \\
 &+ bX_2 - (ax_4 - c)X_3 + aX_4][(4ax_2x_4 - 4cx_2 + a)X_1 \\
 &+ 2(ax_4 - c)X_2 - 2dX_3] = 0, \\
 &-2m(2 + \lambda a) + m[2ax_2(a\partial_2 X_1 + b\partial_4 X_1) + a\partial_2 X_2 + b\partial_4 X_2 \\
 &- (ax_4 - c)\partial_4 X_3 + a\partial_4 X_4] \\
 &- 2a[2bx_2X_1 + bX_2 - (ax_4 - c)X_3 + aX_4][2x_2X_1 + X_2] = 0, \\
 &m(1 + 2\lambda d) + m[(4ax_2x_4 - 4cx_2 + a)\partial_3 X_1 + 2(ax_4 - c)\partial_3 X_2 - 2d\partial_3 X_3] \\
 &+ 2[(2ax_2x_4 - 2cx_2 + \frac{a}{2})X_1 + (ax_4 - c)X_2 - dX_3]^2 = 0, \\
 &m[2ax_2\partial_3 X_1 + a\partial_3 X_2 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_4 X_1 \\
 &- (ax_4 - c)\partial_4 X_2 + d\partial_4 X_3] + a[(4ax_2x_4 - 4cx_2 + a)X_1 \\
 &+ 2(ax_4 - c)X_2 - 2dX_3][2x_2X_1 + X_2] = 0, \\
 &m[2x_2\partial_4 X_1 + \partial_4 X_2] - a[2x_2X_1 + X_2]^2 = 0.
 \end{aligned} \right\} \tag{3.1}$$

The last equation implies that either $2x_2X_1 + X_2 = 0$, or

$$2x_2X_1 + X_2 = \frac{m}{mF_1(x_1, x_2, x_3) - ax_4},$$

for a smooth function F_1 . But the function $2x_2X_1 + X_2$ has to be defined on the whole \mathbb{R}^4 , which implies that $2x_2X_1 + X_2 = 0$. Then (3.1) becomes

$$\left\{ \begin{array}{l} am(a - 4d) \left[a\partial_1X_1 + 4bx_2X_2 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_1X_3 + 2ax_2\partial_1X_4 \right] \\ - a(a - 4d) \left[aX_1 - (4ax_2x_4 - 2cx_2 + \frac{a}{2})X_3 + 2ax_2X_4 \right]^2 \\ + 2m(4abx_2^2 + 48dbx_2^2 - a^2 + 4ad) - m\lambda a(a - 4d)(4bx_2^2 + a) = 0, \\ 8bmx_2(a + 12d) - 4m\lambda a(a - 4d)bx_2 + am(a - 4d) \left[a\partial_1X_4 + a\partial_2X_1 \right. \\ \left. - (ax_4 - c)\partial_1X_3 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_2X_3 + 2ax_2\partial_2X_4 + 2bX_2 \right] \\ - a(a - 4d) \left[aX_1 + (4ax_2x_4 - 4cx_2 + a)X_3 \right. \\ \left. + ax_2X_4 \right] \left[(ax_4 - c)X_3 - aX_4 \right] = 0, \\ m(2 + a\lambda)(4ax_2x_4 - 4cx_2 + a) + am \left[\frac{a}{2}(\partial_1X_1) - 2(ax_4 - c)X_2 \right. \\ \left. - (2ax_2x_4 - 2cx_2 + \frac{a}{2})\partial_3X_3 + d\partial_1X_3 + a\partial_3X_1 + 2ax_2(\partial_3X_4 - X_4) \right] \\ \left. + a \left[aX_1 + 2ax_2X_4 - (2ax_2x_4 - 2cx_2 + \frac{a}{2})X_3 \right] \left[aX_1 - 2dX_3 \right] = 0, \\ -4(2 + a\lambda)x_2 + 2ax_2\partial_4X_4 + a\partial_1X_2 + 2bx_2\partial_4X_2 = 0, \\ 2bm(a + 12d) - am(a - 4d) \left[(ax_4 - c)\partial_2X_3 - a\partial_2X_4 \right] \\ - a(a - 4d) \left[(ax_4 - c)X_3 - aX_4 \right]^2 - am(a - 4d)\lambda b = 0, \\ 2m(2 + a\lambda)(ax_4 - c) + am \left[\frac{a}{2}\partial_2X_1 + a(X_4 - \partial_3X_4) - d\partial_2X_3 \right. \\ \left. + (ax_4 - c)(\partial_2X_2 + \partial_3X_3) \right] + a \left[- (ax_4 - c)X_3 + aX_4 \right] \left[aX_1 - 2dX_3 \right] = 0, \\ -2(2 + a\lambda) + a\partial_4X_4 - (ax_4 - c)\partial_4X_3 = 0, \\ 2m(1 + 2\lambda d) + 2m \left[a\partial_3X_1 - 2d\partial_3X_3 \right] + \left[aX_1 - 2dX_3 \right]^2 = 0, \\ a\partial_4X_1 - 2d\partial_4X_3 = 0. \end{array} \right.$$

Combining the fourth, fifth, seventh, eighth and last equations, we get $\partial_1X_1 = \partial_4X_1 = 0$, $X_4 = \frac{2}{a}(2 + a\lambda)x_4 + F_2(x_1, x_2, x_3)$, for a smooth function F_2 and X_3 is a constant equals $X_3 = \frac{2}{a}(2 + a\lambda)$. Then the first equation of (3.1) implies

$2 + a\lambda = 0$, hence $X_3 = 0$, $\partial_4 X_4 = 0$ and (3.1) is equivalent to

$$\left\{ \begin{array}{l} 2am(a - 4d)x_2[-4bx_2X_1 + a\partial_1X_4] - a(a - 4d)[aX_1 + 2ax_2X_4]^2 \\ + 8bm x_2^2(a + 12d) - 4m\lambda ab(a - 4d)x_2^2 = 0, \\ 8mbx_2(a + 12d) + ma(a - 4d)[a\partial_1X_4 + a\partial_2X_1 + 2ax_2\partial_2X_4 - 4bx_2X_1] \\ - 4m\lambda a(a - 4d)bx_2 + a^3(a - 4d)(X_1 + x_2X_4)X_4 = 0, \\ m[4x_2(ax_4 - c)X_1 + a\partial_3X_1 + 2ax_2(\partial_3X_4 - X_4)] \\ + a^2[X_1 + 2x_2X_4]X_1 = 0, \\ 2bm(a + 12d) + a^2m(a - 4d)\partial_2X_4 - a^3(a - 4d)X_4^2 - abm(a - 4d)\lambda = 0, \\ m[\frac{a}{2}\partial_2X_1 + a(X_4 - \partial_3X_4) - 2x_2(ax_4 - c)\partial_2X_1] + a^2X_4X_1 = 0 \\ 2m(1 + 2\lambda d) + 2am\partial_3X_1 + a^2X_1^2 = 0. \end{array} \right.$$

Since the functions X_i are independents of x_4 and $a \neq 0$, then the third equation implies $X_1 = 0$. We deduce from the last equation that $a - 4d = 0$, which is impossible.

A2: The quasi-Einstein equations for (M, g, X) are given by:

$$\left\{ \begin{array}{l} m\partial_1X_3 + ae^{2\alpha x_4}X_3^2 = 0, \\ m(\partial_1X_2 - \partial_2X_3) + 2ae^{2\alpha x_4}X_2X_3 = 0, \\ md[-a\partial_1X_1 + be^{-2x_4}\partial_1X_3 - a\partial_3X_3 + ce^{-(\alpha+1)x_4}\partial_1X_4 - 2a\alpha X_4] \\ + 2am(3\alpha^2 + \lambda d) + 2adX_3(-ae^{2\alpha x_4}X_1 \\ + be^{2(\alpha-1)x_4}X_3 + 2ce^{(\alpha-1)x_4}X_4) = 0, \\ m[ce^{(\alpha-1)x_4}\partial_1X_3 + d\partial_1X_4 - ae^{2\alpha x_4}\partial_4X_3] \\ + 2ae^{2\alpha x_4}X_3(ce^{(\alpha-1)x_4}X_3 + dX_4) = 0, \\ m(3\alpha^2 + \lambda d) - md(\partial_2X_2 + \alpha X_4) + ade^{2\alpha x_4}X_2^2 = 0, \\ m[a(\partial_3X_2 - \partial_2X_1) + be^{-2x_4}\partial_2X_3 + ce^{-(\alpha+1)x_4}\partial_2X_4] \\ - 2aX_2(-ae^{2\alpha x_4}X_1 + be^{2(\alpha-1)x_4}X_3 + ce^{(\alpha-1)x_4}X_4) = 0, \\ m[ce^{(\alpha-1)x_4}\partial_2X_3 + d\partial_2X_4 + ae^{2\alpha x_4}(\partial_4X_2)] \\ - 2ae^{2\alpha x_4}X_2(2ce^{(\alpha-1)x_4}X_3 + dX_4) = 0, \\ \vdots \end{array} \right.$$

$$\left\{ \begin{array}{l} \vdots \\ md[-ae^{2x_4}\partial_3X_1 + b\partial_3X_3 + ce^{-(\alpha-1)x_4}\partial_3X_4 + 2(\alpha-1)bX_4] \\ -mb(3\alpha^2 - 3\alpha + 2 + \lambda d) \\ -d(-ae^{(\alpha+1)x_4}X_1 + be^{(\alpha-1)x_4}X_3 + cX_4)^2 = 0, \\ md[cd\partial_3X_3 + de^{(1-\alpha)x_4}\partial_3X_4 - e^{(\alpha+1)x_4}(a\partial_4X_1 + c\partial_4X_4) \\ + be^{(\alpha-1)x_4}\partial_4X_3 + (\alpha-1)cX_4] \\ -2mc(3\alpha^2 + \lambda d) - 2d(ce^{(\alpha-1)x_4}X_3 + dX_4)(-ae^{(\alpha+1)x_4}X_1 \\ + be^{(\alpha-1)x_4}X_3 + cX_4) = 0, \\ m(3\alpha^2 + \lambda d) - m[ce^{(\alpha-1)x_4}\partial_4X_3 \\ + d\partial_4X_4] + (ce^{(\alpha-1)x_4}X_3 + dX_4)^2 = 0. \end{array} \right. \quad (3.2)$$

Integrating the first, second, fourth and fifth equations, we find

$$\begin{aligned} X_2 &= F_3(x_2, x_3, x_4), & X_3 &= 0, \\ X_4 &= \frac{1}{dm} [ade^{2\alpha x_4} F_3^2 d - dm\partial_2 F_3 + m(3\alpha^2 + \lambda d)], \end{aligned}$$

for a smooth function F_3 . Then the system (3.2) becomes

$$\left\{ \begin{array}{l} d(\partial_1X_1 + 2\alpha X_4) - 2(3\alpha^2 + \lambda d) = 0, \\ m(3\alpha^2 + \lambda d) - md(\partial_2X_2 + \alpha X_4) + ade^{2\alpha x_4}X_2^2 = 0, \\ m[a(\partial_3X_2 - \partial_2X_1) + ce^{-(\alpha+1)x_4}\partial_2X_4] \\ -2aX_2(-ae^{2\alpha x_4}X_1 + ce^{(\alpha-1)x_4}X_4) = 0, \\ m[d\partial_2X_4 + ae^{2\alpha x_4}\partial_4X_2] - 2ade^{2\alpha x_4}X_2X_4 = 0, \\ md[-ae^{2x_4}\partial_3X_1 + ce^{-(\alpha-1)x_4}\partial_3X_4 + 2(\alpha-1)bX_4] \\ -mb(3\alpha^2 - 3\alpha + 2 + \lambda d) - d(-ae^{(\alpha+1)x_4}X_1 + cX_4)^2 = 0, \\ md[de^{(1-\alpha)x_4}\partial_3X_4 - e^{(\alpha+1)x_4}(a\partial_4X_1 + c\partial_4X_4) + (\alpha-1)cX_4] \\ -2mc(3\alpha^2 + \lambda d) - 2d^2X_4(-ae^{(\alpha+1)x_4}X_1 + cX_4) = 0, \\ m(3\alpha^2 + \lambda d) - md\partial_4X_4 + d^2X_4^2 = 0. \end{array} \right. \quad (3.3)$$

We deduce from the first equation that

$$X_1 = \frac{2}{d}(3\alpha^2 + \lambda d - \alpha d X_4)x_1 + F_4(x_2, x_3, x_4),$$

for a smooth function F_4 . Then the fifth equation of (3.3) implies $3\alpha^2 + \lambda d - \alpha d X_4 = 0$. Thus $\partial_1 X_1 = 0$ and either $\alpha = 0$, or $\alpha \neq 0$ and $X_4 = \frac{1}{\alpha d}(3\alpha^2 + \lambda d)$.

1. If $\alpha \neq 0$ and $X_4 \neq 0$, then (3.3) is equivalent to

$$\left\{ \begin{array}{l} m(\partial_2 X_2) - ae^{2\alpha x_4} X_2^2 = 0, \\ m(\partial_3 X_2 - \partial_2 X_1) + 2X_2(ae^{2\alpha x_4} X_1 - ce^{(\alpha-1)x_4} X_4) = 0, \\ m\alpha\partial_4 X_2 - 2(3\alpha^2 + \lambda d)X_2 = 0, \\ dm[2(\alpha - 1)bX_4 - ae^{2x_4}(\partial_3 X_1)] \\ -bm(3\alpha(\alpha - 1) + 2 + \lambda d) - d(ae^{(\alpha+1)x_4} X_1 - cX_4)^2 = 0, \\ dm[(\alpha - 1)cX_4 - e^{(\alpha+1)x_4}(a\partial_4 X_1)] \\ -2cm(3\alpha^2 + \lambda d) + 2d^2 X_4(ae^{(\alpha+1)x_4} X_1 - cX_4) = 0, \\ (m + 3)\alpha^2 + \lambda d = 0. \end{array} \right. \quad (3.4)$$

The first and third equations imply that $X_2 = 0$. Then

$$\left\{ \begin{array}{l} \partial_2 X_1 = 0, \\ md[2(\alpha - 1)bX_4 - ae^{2x_4}(\partial_3 X_1)] \\ -mb(3\alpha(\alpha - 1) + 2 + \lambda d) - d(ae^{(\alpha+1)x_4} X_1 - cX_4)^2 = 0, \\ md[(\alpha - 1)cX_4 - e^{(\alpha+1)x_4}(a\partial_4 X_1)] \\ -2mc(3\alpha^2 + \lambda d) + 2d^2 X_4(ae^{(\alpha+1)x_4} X_1 - cX_4) = 0, \\ (m + 3)\alpha^2 + \lambda d = 0. \end{array} \right.$$

The last system has a non-zero solution if, and only if,

$$X_1 = -\frac{\alpha cm}{ad} e^{-(\alpha+1)x_4}.$$

However in that case, the third equation implies $X_4 = 0$, which is a contradiction.

2. If $X_4 = 0$, then (3.3) is equivalent to

$$\begin{cases} m\partial_2 X_2 - ae^{2\alpha x_4} X_2^2 = 0, \\ m(\partial_3 X_2 - \partial_2 X_1) + 2aX_2 e^{2\alpha x_4} X_1 = 0, \\ \partial_4 X_2 = 0, \\ adme^{2x_4} \partial_3 X_1 + bm(-3\alpha + 2) + a^2 de^{2(\alpha+1)x_4} X_1^2 = 0, \\ \partial_4 X_1 = 0, \end{cases} \quad (3.5)$$

then $X_1 = X_2 = 0$.

3. If $\alpha = 0$, then $\lambda = 0$ and since X_2 has to be defined everywhere, the first equation of (3.3) implies $X_2 = 0$ and the last one entails $X_4 = 0$. Hence (3.3) is equivalent to

$$\begin{cases} \partial_2 X_1 = \partial_4 X_1 = 0, \\ adme^{2x_4} \partial_3 X_1 + 2mb + a^2 de^{2x_4} X_1^2 = 0, \end{cases} \quad (3.6)$$

which implies $b = 0$ and $X_1 = 0$.

A3 : We denote $f(x_4) = \cos(x_4)$ for $\varepsilon = 1$ and $f(x_4) = \cosh(x_4)$ for $\varepsilon = -1$. Then the quasi-Einstein equations are described as follows:

$$\begin{cases} m\partial_1 X_4 - ae^{2x_3} X_4 = 0, \\ m[f(x_4)^2 \partial_1 X_2 + \partial_2 X_4] - 2ae^{2x_3} f(x_4)^2 X_2 X_4 = 0, \\ m[b\partial_1 X_3 + c\partial_1 X_4 + ae^{2x_3} \partial_3 X_4] - 2ae^{2x_3} X_4 (bX_3 + cX_4) = 0, \\ 2ame^{2x_3} (3 + b\lambda) + bm[ae^{2x_3} (\partial_1 X_1 + \partial_4 X_4) + 2ae^{2x_3} X_3 + c\partial_1 X_3 + d\partial_1 X_4] \\ - 2abe^{2x_3} X_4 [ae^{2x_3} X_1 + cX_3 + dX_4] = 0, \\ m(3 + b\lambda)f(x_4) - 2bm[f(x_4)(\partial_2 X_2 + X_3) \\ - h(x_4)X_4] + abe^{2x_3} f(x_4)^3 X_2^2 = 0, \\ m[b\partial_2 X_3 + c\partial_2 X_4 + ae^{2x_3} f(x_4)^2 \partial_3 X_2] - 2ae^{2x_3} f(x_4)^2 X_2 (bX_3 + cX_4) = 0, \\ m[ae^{2x_3} \partial_2 X_1 + d\partial_2 X_4 + ae^{2x_3} f(x_4)^2 \partial_4 X_2 + c\partial_2 X_3] \\ - 2ae^{2x_3} f(x_4)^2 X_2 (ae^{2x_3} X_1 + cX_3 + dX_4) = 0, \\ \vdots \end{cases}$$

$$\left\{ \begin{array}{l} \vdots \\ m(3 + b\lambda) - m[b\partial_3 X_3 + c\partial_3 X_4] + 2(bX_3 + cX_4)^2 = 0, \\ -2cm(3 + b\lambda) + mb[ae^{2x_3}\partial_3 X_1 + c\partial_3 X_3 + d\partial_3 X_4 + b\partial_4 X_3 + c\partial_4 X_4] \\ -2b(bX_3 + cX_4)(ae^{2x_3} X_1 + cX_3 + dX_4) = 0, \\ m(\varepsilon b - 2d - \lambda bd) + bm[ae^{2x_3}\partial_4 X_1 + c\partial_4 X_3 + d\partial_4 X_4] \\ -b(ae^{2x_3} X_1 + cX_3 + dX_4)^2 = 0, \end{array} \right.$$

where $h(x_4) = \sin(x_4)$ for $\varepsilon = 1$ and $h(x_4) = \sinh(x_4)$ for $\varepsilon = -1$.

Integrating the first, third and eighth equations, we find $X_4 = 0$ and $\partial_1 X_3 = 0$. Then the system becomes

$$\left\{ \begin{array}{l} \partial_1 X_2 = 0, \\ 2(3 + b\lambda) - b[\partial_1 X_1 + 2X_3] = 0, \\ 2m(3 + b\lambda) - 2mb(\partial_2 X_2 + X_3) + abe^{2x_3} f(x_4)^2 X_2^2 = 0, \\ m[b\partial_2 X_3 + ae^{2x_3} f(x_4)^2 \partial_3 X_2] - 2abe^{2x_3} f(x_4)^2 X_2 X_3 = 0, \\ m[ae^{2x_3} \partial_2 X_1 + ae^{2x_3} f(x_4)^2 \partial_4 X_2 + c\partial_2 X_3] \\ -2ae^{2x_3} f(x_4)^2 X_2 (ae^{2x_3} X_1 + cX_3) = 0, \\ -2mc(3 + b\lambda) + mb[ae^{2x_3} \partial_3 X_1 + c\partial_3 X_3 + b\partial_4 X_3] \\ -2b^2 X_3 (ae^{2x_3} X_1 + cX_3) = 0, \\ m(\varepsilon b - 2d - \lambda bd) + mb[ae^{2x_3} \partial_4 X_1 + c\partial_4 X_3] - b(ae^{2x_3} X_1 + cX_3)^2 = 0. \end{array} \right.$$

We deduce from the second equation of the last system that

$$X_1 = \frac{2}{b}[b(\lambda - X_3) + 3]x_1 + F_5(x_2, x_3, x_4),$$

for a smooth function F_5 . Since $\partial_1 X_3 = 0$, then the last equation of the system implies $b\lambda + 3 - bX_3 = 0$, which implies $X_3 = 0$ and $b\lambda + 3 = 0$, then $X_1 = X_2 = 0$ and $d = -\varepsilon b$.

A4: Since the function $2x_2 X_1 + X_2$ has to be defined on the whole \mathbb{R}^4 , the (33)-component implies $2x_2 X_1 + X_2 = 0$ and the quasi-Einstein equations are given as:

$$\left\{ \begin{array}{l}
2am[2(x_4^2 + a)\partial_1 X_1 + ax_2(4 + x_4^2)\partial_1 X_3 + a(1 + 2x_2x_3)\partial_1 X_4 \\
+ ax_4 X_4 + 8bx_2 X_2] - 2m(3ax_4^2 + 6a + 64bx_2^2) - 2am\lambda(ax_4^2 + 8bx_2^2 + 2a) \\
- a^3[(2 + x_4^2)X_1 + x_2(4 + x_4^2)X_3 + (1 + 2x_2x_3)x_4 X_4]^2 = 0, \\
-16mbx_2(8 + a\lambda) + am[a(4 + x_4^2)(2x_2\partial_2 X_3 + \partial_1 X_3) + 2ax_3x_4\partial_1 X_4 \\
+ 8bX_2 + 2a(2x_2x_4 + 1)x_4\partial_2 X_4 + 2a(x_4^2 + 2)\partial_2 X_1] - a^3[(x_4^2 + 2)X_1 \\
+ x_2(4 + x_4^2)X_3 + (1 + 2x_2x_3)x_4] [(4 + x_4^2)X_3 + 2x_3x_4 X_4] = 0, \\
-2x_2(x_4^2 + 4)(3 + a\lambda) + a[(4 + x_4^2)(x_2\partial_3 X_3 + X_2) \\
+ (x_4^2 + 2)\partial_3 X_1 + (1 + 2x_2x_3)x_4\partial_3 X_4 + 2x_2x_4 X_4] = 0, \\
-2m(3 + a\lambda)x_4(1 + 2x_2x_3) + m[ax_4(\partial_1 X_1 + \partial_4 X_4) + a\partial_1 X_4 + 2ax_3x_4 X_2 \\
+ a(x_4^2 + 2)(\partial_4 X_1) + 2ax_2x_4 X_3 + ax_2(4 + x_4^2)\partial_4 X_3 + 2(2x_2x_3 + 1)X_4] \\
- a^2[(x_4^2 + 2)X_1 + x_2(4 + x_4^2)X_3 + (1 + 2x_2x_3)x_4] (x_4 X_1 + X_4) = 0, \\
4bm(8 + a\lambda) - am[a(4 + x_4^2)\partial_2 X_3 + 2ax_3x_4\partial_2 X_4] \\
+ a^2[(4 + x_4^2)X_3 + 2x_3x_4 X_4]^2 = 0, \\
2(3 + a\lambda)(4 + x_4^2) - a(4 + x_4^2)\partial_3 X_3 - 2ax_3x_4\partial_3 X_4 - 2ax_2 X_4 = 0, \\
-am[2x_3x_4(\partial_2 X_1 + \partial_4 X_4) + 2(\partial_2 X_4) + (4 + x_4^2)\partial_4 X_3 + 4x_4 X_3 + 2x_3 X_4] \\
+ 4m(3 + a\lambda)x_3x_4 + a^2[(4 + x_4^2)X_3 + 2ax_3x_4 X_4] (x_4 X_1 + X_4) = 0, \\
2x_4\partial_3 X_1 + (x_4^2 + 4)\partial_3 X_4 = 0, \\
-2m(3 + a\lambda) + 2am[x_4\partial_4 X_1 + \partial_4 X_4] - a^2(x_4 X_1 + X_4)^2 = 0.
\end{array} \right. \quad (3.7)$$

Integrating the sixth, eighth and ninth equations, we find $\partial_3 X_1 = \partial_3 X_4 = 0$. Then the third and sixth equations of (3.7) become, respectively,

$$\left\{ \begin{array}{l}
2(3 + a\lambda)(4 + x_4^2) - a(4 + x_4^2)(\partial_3 X_3) - 2a(4 + x_4^2)X_1 - 2ax_4 X_4 = 0, \\
2(3 + a\lambda)(4 + x_4^2) - a(4 + x_4^2)(\partial_3 X_3) - 2ax_4 X_4 = 0.
\end{array} \right.$$

Taking the difference of these two equations, we find $X_1 = 0$ and, consequently, $X_2 = 0$. Hence

$$X_3 = -\frac{2x_4x_3}{4 + x_4^2}X_4 + \frac{2}{a}(3 + a\lambda)x_3 + F_6(x_1, x_2, x_4),$$

for a smooth function F_6 .

The coefficient of x_3^2 in the fifth equation implies $\lambda = -3/a$, and since X_4 has to be defined everywhere, then the last equation of (3.7) entails $X_4 = 0$. Then the coefficient of x_3 in the second equation of (3.7) implies $X_3 = 0$. Thus $X = 0$ and $A4$ is necessarily an Einstein manifold.

A5: Locally the quasi-Einstein equations for (M, g, X) are described by

$$\left\{ \begin{aligned}
 &8mx_2(x_2\partial_1X_4 - x_4\partial_1X_2) - a(x_4X_2 - x_2X_4)^2 = 0, \\
 &8mx_2^2x_4(12 + a\lambda) + 4am[x_2^2\partial_2X_4 - x_2x_3\partial_1X_3 - x_1x_2\partial_1X_4 + x_2x_4\partial_1X_1 \\
 &\quad + 2x_1x_4\partial_1X_2 + x_3^2\partial_1X_2 - x_2x_4\partial_2X_2 + x_4X_2 - x_2X_4 + 2\partial_1X_2] - a^2(x_4X_2 \\
 &\quad - x_2X_4)[x_2x_4X_1 - a(2 + 2x_1x_4 + x_3^2)X_2 + x_2x_3X_3 + x_1x_2X_4] = 0, \\
 &4mx_2[x_2\partial_3X_4 + x_2\partial_1X_3 - x_3\partial_1X_2 - x_4\partial_3X_2] \\
 &\quad - a(x_4X_2 - x_2X_4)(x_3X_2 - x_2X_3) = 0, \\
 &4amx_2[x_2(\partial_4X_4 + \partial_1X_1) - x_1\partial_1X_2 - x_4\partial_4X_2] \\
 &\quad + a^2(x_4X_2 - x_2X_4)(x_2X_1 - x_1X_2) - 8mx_2(12 + a\lambda) = 0, \\
 &-96m(3x_1x_4 + 2 + x_3^2)x_2^2 - 8amx_2[x_2^2(x_1\partial_2X_4 + x_3\partial_2X_3) - 2x_1x_2x_4\partial_2X_2 \\
 &\quad - x_2x_3^2\partial_2X_2 + x_2^2x_4\partial_2X_1 + (2x_1x_4 + x_3^3)X_2 - x_1x_2X_4 - x_2x_4X_1 \\
 &\quad - x_2x_3X_3 - 2x_2\partial_2X_2 + 2X_2] \\
 &\quad - a^2[x_2(x_4X_1 + x_3X_3 + x_1X_4) - (2 + 2x_1x_4 + x_3^2)X_2]^2 \\
 &\quad - 8a\lambda m(2 + 2x_1x_4 + x_3^2)x_2^2 = 0, \\
 &96mx_3x_2^2 + 32am\lambda x_2^2x_3 - 4amx_2[x_1x_2\partial_3X_4 + x_3x_2\partial_3X_3 - x_2^2\partial_2X_3 \\
 &\quad - (2x_1x_4 + x_3^2 + 2)\partial_3X_2 + x_2x_3\partial_2X_2 + x_2x_4\partial_3X_1 - x_3X_2 + X_3] \\
 &\quad - a^2[x_2(x_4X_1 + x_3X_3 + x_1X_4) - (2 + 2x_1x_4 + x_3^2)X_2](x_3X_2 - x_2X_3) = 0, \\
 &96mx_1x_2^2 - a\lambda mx_2^2x_3 - 4amx_2[x_1x_2\partial_4X_4 + x_2x_3\partial_4X_3 - x_2^2\partial_2X_1 \\
 &\quad + x_2x_4\partial_4X_1 + x_1x_2\partial_2X_2 - 2x_1x_4\partial_4X_2 - x_3^2\partial_4X_2 - x_1X_2 + x_2X_1 - 2\partial_4X_2] \\
 &\quad + a[x_4x_2X_1 - (2 + 2x_1x_4 + x_3^2)X_2 \\
 &\quad + x_3x_2X_3 + x_1x_2X_4](x_2X_1 - x_1X_2) = 0, \\
 &12mx_2 - am(x_2\partial_3X_3 - x_3\partial_3X_2) - a(x_3X_2 - x_2X_3)^2 + a\lambda mx_2 = 0, \\
 &4mx_2[x_2(\partial_4X_3 + \partial_3X_1) - x_1\partial_3X_2 - x_3\partial_4X_2] \\
 &\quad + 2a(x_3X_2 - x_2X_3)(x_2X_1 - x_1X_2) = 0, \\
 &8mx_2(x_1\partial_4X_2 - x_2\partial_4X_1) + a(x_2X_1 - x_1X_2)^2 = 0.
 \end{aligned} \right. \tag{3.8}$$

The last equation implies $x_2X_1 - x_1X_2 = 0$, then substituting into the first equation of (3.8), we find that necessarily $x_2X_4 - x_4X_2 = 0$. Consequently, the fourth equation implies $12 + a\lambda = 0$ and the eighth gives $x_2X_3 - x_3X_2 = 0$. Then (3.8) becomes

$$\begin{cases} \partial_1X_2 = \partial_3X_2 = \partial_4X_2 = 0, \\ 4mx_2[\partial_2X_2 - X_2] - aX_2^2 = 0. \end{cases}$$

Then either $X_2 = 0$ or $X_2 = \frac{4me^{x_2}}{aF(x_2) + 4c_1m}$, where $F(x_2) = \int_1^\infty \frac{e^{tx_2}}{t} dt$ and c_1 is a real constant, but this function is not defined when x_2 takes positive values. Hence $X_2 = 0$ and consequently $X = 0$ and A5 is an Einstein manifold.

B1: The (44)-component implies

$$m(2x_2(\partial_4X_1) + \partial_4X_2) - a(2x_2X_1 + X_2)^2 = 0,$$

and since $2x_2X_1 + X_2 = 0$ is defined on the whole \mathbb{R}^4 , then $2x_2X_1 + X_2 = 0$.

Hence the structure (M, g, X) is quasi-Einstein if, and only if, the following system is satisfied

$$\left\{ \begin{array}{l} 3m[d^2(x_3 + 2x_2x_4)^2 + d((24b + 8cx_4)x_2^2 + 4cx_2x_3 + 2ax_3) - 20c^2x_2^2] \\ - 2\lambda a^2m(d(x_3^2 + 4x_2x_3x_4 + 4x_2^2x_4^2) + 4cx_2x_3 + 8cx_2^2x_4 + 2ax_3 + 4bx_2^2) \\ + m[2ax_2\partial_1X_4 + (2dx_2x_4 + 2cx_2 + dx_3 + a)\partial_1X_3 + (dx_4 + c)x_3\partial_1X_2 \\ + (4dx_2x_4 + cx_2 + dx_3 + 2a)x_3\partial_1X_1 + 2(2x_2(dx_4 + c) + dx_3 + a)X_3 \\ + 4(2dx_2x_4^2 + 4cx_2x_4 + dx_3x_4 + 2bx_2 + cx_3)X_2 \\ + 4x_2(2x_2(dx_4 + c) + dx_3)X_4] - 2a^2[(dx_3 + 2dx_2x_4 + 2cx_2 + 2a)x_3X_1 \\ + (dx_3 + 2dx_2x_4 + 2cx_2 + a)X_3 + 2ax_2X_4]^2 = 0, \\ m(3d - 2a^2\lambda)(4cx_2x_4 + dx_3x_4 + 2dx_2x_4^2 + cx_3) \\ + 3m(12bdx_2 - 10c^2x_2) - 4a^2bm\lambda x_2 \\ + a^2m[2ax_2\partial_2X_4 + a\partial_1X_4 + (2x_2(dx_4 + c) + dx_3 + a)\partial_2X_3 \\ + (dx_4 + c)\partial_1X_3 + (2dx_2x_4 + 2cx_2 + dx_3 + 2a)x_3\partial_2X_1 + (dx_4 + c)x_3\partial_1X_1] \\ - a^2[(d(x_3 + 2x_2x_4) + 2cx_2 + 2a)x_3X_1 + (d(x_3 + 2x_2x_4) + 2cx_2 + a)X_3 \\ + 2ax_2X_4][(dx_4 + c)(x_3X_1 + X_3) + aX_4] = 0, \\ \vdots \end{array} \right.$$

$$\left\{ \begin{array}{l}
 \vdots \\
 (3d - 2a^2m\lambda)(dx_3 + 2dx_2x_4 + 2cx_2 + a) + a^2m[(2dx_2x_4 + 2cx_2 + dx_3 \\
 + a)(\partial_3X_3) + 2ax_2\partial_3X_4 + d\partial_1X_3 + (2dx_2x_4 + 2cx_2 + dx_3 + 2a)x_3\partial_3X_1 \\
 + (dx_3 + a)\partial_1X_1 + 2(dx_4 + c)X_2 + dX_3 + 2dx_2X_4] - 2a^2[(d(x_3 + 2x_2x_4) \\
 + 2(cx_2 + a))x_3X_1 + (d(x_3 + 2x_2x_4) + 2cx_2 + a)X_3 \\
 + 2ax_2X_4][(dx_3 + a)X_1 + dX_3] = 0, \\
 2(3d - 2a^2\lambda)x_2 + a[2ax_2(\partial_4X_4) + (2dx_2x_4 + 2cx_2 + dx_3 + a)\partial_4X_3 \\
 + (2dx_2x_4 + 2cx_2 + dx_3 + 2a)x_3\partial_4X_1 + 2aX_2] = 0, \\
 3(d^2x_4^2 + 2cdx_4 - 5c^2 + 6bd) + 2a^2m[\partial_2X_4 + (dx_4 + c)\partial_2X_3 \\
 + (dx_4 + c)(x_3\partial_2X_1 + X_4)] - 4a^2[(dx_4 + c)(x_3X_1 + X_3) + aX_4]^2 \\
 - 2a^2m\lambda(dx_4^2 + 2cx_4 + b) = 0, \\
 m(3d - 2a^2\lambda)(dx_4 + c) + a^2m[a\partial_3X_4 + d\partial_2X_3 + (dx_4 + c)(x_3\partial_3X_1 \\
 + \partial_3X_3) + dX_4 + (dx_4 + a)\partial_2X_1] - 2a^2[(dx_4 + c)x_3X_1X_3 \\
 + aX_4][(dx_3 + a)X_1 + dX_3] = 0, \\
 3d - 2\lambda a^2 + a[a\partial_4X_4 + (dx_4 + c)(x_3\partial_4X_1 + \partial_4X_3)] = 0, \\
 d(3d - 2a^2\lambda) + 2a^2[d\partial_3X_3 + (dx_3 + a)\partial_3X_1] \\
 - [(dx_3 + a)X_1 + dX_3]^2 = 0, \\
 d\partial_4X_3 + (dx_3 + a)\partial_4X_1 = 0.
 \end{array} \right. \tag{3.9}$$

Integrating the seventh, eighth and ninth equations, we find that necessarily $\partial_4X_1 = \partial_4X_3 = 0$. Then the fourth and seventh equations of (3.9) become, respectively,

$$\begin{cases}
 3d - 2a^2\lambda + a^2(\partial_4X_4) - 2a^2X_1 = 0, \\
 3d - 2a^2\lambda + a^2\partial_4X_4 = 0.
 \end{cases}$$

Hence $X_1 = X_2 = 0$ and $\partial_4X_4 = \frac{1}{a^2}(2a^2\lambda - 3d)$, thus (3.9) is equivalent to

$$\left\{ \begin{array}{l} 3[d^2(x_3 + 2x_2x_4)^2 + d((24b + 8cx_4)x_2^2 + 4cx_2x_3 + 2ax_3) - 20c^2x_2^2] \\ -2a^2\lambda(d(x_3^2 + 4x_2x_3x_4 + 4x_2^2x_4^2) + 4cx_2x_3 + 8cx_2^2x_4 + 2ax_3 + 4bx_2^2) \\ -2a^2[(d(x_3 + 2x_2x_4) + 2cx_2 + a)X_3 + 2ax_2X_4]^2 \\ +2a^2m[2ax_2(\partial_1X_4) + (2dx_2x_4 + 2cx_2 + dx_3 + a)(\partial_1X_3) \\ +2(2x_2(dx_4 + c) + dx_3 + a)X_3 + 4x_2(2x_2(dx_4 + c) + dx_3)X_4] = 0, \\ 3(4cdx_2x_4 + d^2x_3x_4 + 2d^2x_2x_4^2 + cdx_3 + 12bdx_2 - 10c^2x_2) \\ +a^2m[2ax_2(\partial_2X_4) + a(\partial_1X_4) + (2x_2(dx_4 + c) + dx_3 + a)(\partial_2X_3) \\ +(dx_4 + c)(\partial_1X_3)] - a^2[(d(x_3 + 2x_2x_4) + 2cx_2 + a)X_3 \\ +2ax_2X_4][(dx_4 + c)X_3 + aX_4] \\ -2a^2m\lambda[d(x_3x_4 + 2x_2x_4^2) + 4cx_2x_4 + cx_3 + 2bx_2] = 0, \\ (3d - 2a^2m\lambda)(dx_3 + 2dx_2x_4 + 2cx_2 + a) + a^2m[dX_3 \\ +(2dx_2x_4 + 2cx_2 + dx_3 + a)(\partial_3X_3) + d(\partial_1X_3) + 2x_2(dX_4 + a\partial_3X_4)] \\ -2a^2d[(d(x_3 + 2x_2x_4) + 2cx_2 + a)X_3 + 2ax_2X_4]X_3 = 0, \\ 3(d^2x_4^2 + 2cdx_4 - 5c^2 + 6bd) + 2a^2m[\partial_2X_4 + (dx_4 + c)\partial_2X_3 \\ +(dx_4 + c)X_4] - 4a^2[(dx_4 + c)X_3 + aX_4]^2 = 2a^2m\lambda(dx_4^2 + 2cx_4 + b), \\ m(3d - 2a^2\lambda)(dx_4 + c) + a^2m[a(\partial_3X_4) + d(\partial_2X_3) \\ +(dx_4 + c)\partial_3X_3 + dX_4] - 2a^2d[(dx_4 + c)X_3 + aX_4]X_3 = 0, \\ d(3d - 2a^2\lambda) + 2a^2d(\partial_3X_3) - d^2X_3^2 = 0. \end{array} \right.$$

Taking the coefficient of x_4 in the fifth equation, we obtain

$$adm\partial_3X_3 - 2d[adX_3 + 2\lambda a^2 - 3d]X_3 = 0. \quad (3.10)$$

Substituting into the last equation, we find

$$dm(3d - 2a^2\lambda) + 4ad(2\lambda a^2 - 3d)X_3 + (4a^2 - m)d^2X_3^2 = 0.$$

Hence either X_3 is a constant and $d = 0$ or $m = 4a^2$ and $3d - 2a^2\lambda = 0$.

Suppose $d = 0$, then the third and fifth equations of the last system become, respectively,

$$\begin{cases} 2(2cx_2 + a) - (2cx_2 + a)(\partial_3X_3) - 2ax_2(\partial_3X_4) = 0, \\ 2c\lambda - a\partial_3X_4 - c\partial_3X_3 = 0, \end{cases}$$

that implies $a = 0$, which is a contradiction, hence $d \neq 0$.

Let us assume that $m = 4a^2$ and $3d - 2a^2\lambda = 0$, then $\partial_4 X_4 = 0$. Since X_3 has to be defined everywhere, then the last equation implies $X_3 = 0$. Consequently, the fifth equation implies $X_4 = 0$, hence (M, g) is Einstein.

Suppose now that X_3 is a constant, then, comparing the last equation of the system and (3.10), we find $X_3 = 0$, hence $3d - 2a^2\lambda = 0$. Then we obtain the same result as the previous case.

B2: Since the function $2x_2X_1 + X_2$ has to be everywhere defined, then the (33)-component implies $2x_2X_1 + X_2 = 0$, and consequently the quasi-Einstein equations are given as follows

$$\left\{ \begin{array}{l} 2m(3ax_4^2 - 6a - 64bx_2^2) - 2am[a(2x_2x_3 + 1)x_4\partial_1X_4 + ax_2(x_4^2 - 4)\partial_1X_3 \\ - (2a - ax_4^2)\partial_4X_1 - 16bx_2^2X_1 + ax_4X_4] - 2am\lambda(2a - ax_4^2 + 8bx_2^2) \\ - a^3[(2 - x_4^2)X_1 - x_2(x_4^2 - 4)X_3 - (1 + 2x_2x_3)x_4X_4]^2 = 0, \\ -16bx_2(8 + a\lambda) + ma[-2a(1 + 2x_2x_3)x_4\partial_2X_4 - 2ax_3x_4\partial_1X_4 \\ - a(x_4^2 - 4)(2x_2\partial_2X_3 + \partial_1X_3) + (4a - 2ax_4^2 + 16bx_2^2)\partial_2X_1 \\ + 8bx_2\partial_1X_1 + 8bX_2] + a^2[2a(2 - x_4^2)X_1 - 2ax_2(x_4^2 - 4)X_3 \\ - 2(1 + 2x_2x_3)x_4][(x_4^2 - 4)X_3 + x_3x_4X_4] = 0, \\ 4x_2(x_4^2 - 4)(3 + a\lambda) - [2a(1 + 2x_2x_3)x_4\partial_3X_4 \\ + a(x_4^2 - 4)(2x_2\partial_3X_3 + \partial_3X_1 + X_2) + 4ax_2x_4X_4] = 0, \\ 2m(1 + 2x_2x_3)x_4(3 + a\lambda) + m[-a\partial_1X_4 - ax_2(x_4^2 - 4)\partial_4X_3 \\ + (2a - ax_4^2)\partial_4X_1 - a(1 + 2x_2x_3)(x_4\partial_4X_4 + X_4) - ax_4\partial_1X_1 \\ - 2ax_4(x_3X_2 + x_2X_3)] + a^2[(2 - x_4^2)X_1 - (x_4^2 - 4)x_2X_3 \\ - (1 + 2x_2x_3)x_4X_4](x_4X_1 + X_4) = 0, \\ 4mbx_2(8 + a\lambda) + ma^2[2x_3x_4\partial_2X_4 + (x_4^2 - 4)\partial_2X_3] \\ + a^3[(x_4^2 - 4)X_3 + x_3x_4X_4]^2 = 0, \\ 6m(x_4^2 - 4) - am[2x_3x_4\partial_3X_4 + (x_4^2 - 4)\partial_3X_3 + 2x_4X_4] \\ - 4m\lambda a(1 + 2x_2x_3) = 0, \\ -am[2x_3x_4\partial_4X_4 + 2\partial_2X_4 + (x_4^2 - 4)\partial_4X_3 + 2x_4\partial_2X_1 + 2x_4X_3 + x_3X_4] \\ + 4mx_3x_4(3 + a\lambda) - 2a^2[(x_4^2 - 4)X_3 + x_3x_4X_4](x_4X_1 + X_4) = 0, \\ \vdots \end{array} \right.$$

$$\left\{ \begin{array}{l} \vdots \\ \partial_3 X_4 + x_4 \partial_3 X_1 = 0, \\ 2m(3 + a\lambda) - 2am[\partial_4 X_4 + x_4 \partial_4 X_1] - a^2(x_4 X_1 + X_4)^2 = 0. \end{array} \right. \quad (3.11)$$

Combining the fifth, eighth and ninth equations of the last system, we find

$$\partial_3 X_1 = \partial_3 X_4 = 0.$$

Hence the third and sixth equations become, respectively,

$$\left\{ \begin{array}{l} 2(x_4^2 - 4)(3 + a\lambda) - a(x_4^2 - 4)(\partial_3 X_3) + a(x_4^2 - 4)X_1 - 2ax_4 X_4 = 0, \\ 6(x_4^2 - 4) - a(x_4^2 - 4)(\partial_3 X_3) - 2ax_4 X_4 - 4\lambda a(1 + 2x_2 x_3) = 0. \end{array} \right. \quad (3.12)$$

Comparing these two equations, we find $\lambda = 0$ and $X_1 = X_2 = 0$. Then by substituting into the first equation we find

$$a(x_4^2 - 4)X_3 = [6(x_4^2 - 4) - 2ax_4 X_4]x_3 + F(x_1, x_2, x_4),$$

which gives a contradiction with the first equation of (3.11).

B3: The quasi-Einstein equations for (M, g, X) are given as follows

$$\left\{ \begin{array}{l} m(x_3 \partial_1 X_2 - \partial_1 X_3) + ae^{-x_2}(x_3 X_2 - X_3)^2 = 0, \\ m[a\partial_1 X_4 + ae^{-x_2} \partial_2 X_3 - 2bx_3 \partial_1 X_3 - ax_3 e^{-x_2} \partial_2 X_2 - ax_3 e^{-x_2} \partial_1 X_1 \\ + ax_3 e^{-x_2} X_2 + 4bx_3^2 \partial_1 X_2 + ae^{-x_2} X_3 + 2ax_4 \partial_1 X_2] \\ + 2ae^{-x_2}(x_3 X_2 - X_3)[-ax_3 e^{-x_2} X_1 \\ + 2(2bx_3^2 - ax_4)X_2 - 2bx_3 X_3 + aX_4] + 2am\lambda x_3 e^{-x_2} = 0, \\ m[ae^{-x_2} \partial_3 X_3 + b\partial_1 X_3 - ax_3 e^{-x_2} \partial_3 X_2 + ae^{-x_2} \partial_1 X_1 - ae^{-x_2} X_2 \\ - 2bx_3 \partial_1 X_2] + 2ae^{-x_2}(x_3 X_2 - X_3)[ae^{-x_2} X_1 \\ - 2bx_3 X_2 + bX_3] - a\lambda m e^{-x_2} = 0, \\ m[-ax_3 e^{-x_2} \partial_4 X_2 + e^{-x_2} \partial_4 X_3 + \partial_1 X_2] + 2a^2 e^{-x_2}(x_3 X_2 - X_3)X_2 = 0, \\ m[2a\partial_2 X_4 - 4bx_3 \partial_2 X_3 + 8bx_3^2 \partial_2 X_2 - 2ax_3 e^{-x_2} \partial_2 X_1 - 4ax_4 \partial_2 X_2 \\ + 8bx_3 X_3 - 2aX_4] - 2[ax_3 e^{-x_2} X_1 - 2(2bx_3^2 - ax_4)X_2 + 2bx_3 X_3 - aX_4]^2 \\ - 2m\lambda(2bx_3^2 - ax_4) = 0, \\ \vdots \end{array} \right.$$

$$\left\{ \begin{array}{l} \vdots \\ m[a\partial_3 X_4 - 2bx_3(\partial_3 X_3 + \partial_2 X_2) + b\partial_2 X_3 + 2(2bx_3^2 - ax_4)\partial_3 X_2 \\ -ae^{-x_2}(x_3\partial_3 X_1 - \partial_2 X_1) - 2bX_3] - 2[ax_3e^{-x_2}X_1 - 2(2bx_3^2 - ax_4)X_2 \\ + 2bx_3X_3 - aX_4][ae^{-x_2}X_1 - 2bx_3X_2 + bX_3] + 2bm\lambda x_3 = 0, \\ 4b\lambda mx_3 + m[a\partial_4 X_4 - 2bx_3\partial_4 X_3 + 4bx_3^2\partial_4 X_2 - ax_3e^{-x_2}\partial_4 X_1 \\ - 2ax_4\partial_4 X_2 + a\partial_2 X_2] - 2[ax_3e^{-x_2}X_1 - 2(2bx_3^2 - ax_4)X_2 \\ + 2bx_3X_3 - aX_4][ae^{-x_2}X_1 - 2bx_3X_2 + bX_3] = 0, \\ m[2b\partial_3 X_3 - 4bx_3\partial_3 X_2 + 2ae^{-x_2}\partial_3 X_1] \\ - 2[ae^{-x_2}X_1 - 2bx_3X_2 + bX_3]^2 - 2bm\lambda = 0, \\ m[b\partial_4 X_3 - 2bx_3\partial_4 X_2 + ae^{-x_2}\partial_4 X_1 + a\partial_3 X_2] \\ - 2a[ae^{-x_2}X_1 - 2bx_3X_2 + bX_3]X_4 = 0, \\ m\partial_4 X_2 - a^2X_2^2 = 0. \end{array} \right. \tag{3.13}$$

The last equation of (3.13) implies $X_2 = 0$, and since X_3 has to be defined on the whole \mathbb{R}^4 , then the first equation implies $X_3 = 0$. Consequently (3.13) becomes

$$\left\{ \begin{array}{l} \partial_1 X_4 + \lambda x_3 e^{-x_2} = 0, \\ \partial_1 X_1 - \lambda = 0, \\ am[\partial_2 X_4 - x_3 e^{-x_2} \partial_2 X_1 - X_4] - a(x_3 e^{-x_2} X_1 - X_4)^2 \\ - m\lambda(2bx_3^2 - ax_4) = 0, \\ am[\partial_3 X_4 - e^{-x_2}(x_3\partial_3 X_1 - \partial_2 X_1)] - 2a^2 e^{-x_2} X_1 [x_3 e^{-x_2} X_1 - X_4] \\ + 2bm\lambda x_3 = 0, \\ am[\partial_4 X_4 - x_3 e^{-x_2} \partial_4 X_1] - 2a e^{-x_2} X_1 [ax_3 e^{-x_2} X_1 - aX_4] + 4b\lambda mx_3 = 0, \\ ame^{-x_2} \partial_3 X_1 - a^2 e^{-2x_2} X_1^2 - bm\lambda = 0, \\ m\partial_4 X_1 - 2aX_1 X_4 = 0. \end{array} \right. \tag{3.14}$$

But that system does not admit any solution unless $\lambda = 0$ and $X_1 = X_4 = 0$.

Remark 3.1. The fact that the homogeneous spaces $A5$ and $B3$ are Einstein manifolds has already been established (see Proposition 2.3). For completeness, we provided a direct proof of this result in the course of the proof of the Main Theorem.

CONCLUSION AND PERSPECTIVES

In this work, we have established that the quasi-Einstein condition on four-dimensional non-reductive homogeneous spaces is equivalent to the Einstein condition. This result highlights a rigidity phenomenon: within this class of manifolds, the generalization from Einstein to quasi-Einstein structures introduces no genuinely new solutions.

Our findings resonate with other known instances where quasi-Einstein metrics reduce to Einstein metrics. For example, in certain warped product settings [7] or under curvature restrictions [11, 10], the quasi-Einstein equation similarly collapses to the Einstein one. By situating our result within this broader context, we reinforce the emerging picture that quasi-Einstein geometry often exhibits strong rigidity, depending on the underlying geometric structure.

Beyond its intrinsic interest, this equivalence has implications for related areas. Since quasi-Einstein metrics naturally appear as fixed points of weighted Ricci flows, our result shows that, in this setting, the flow admits no new fixed points beyond the Einstein case (cf. [12]). Likewise, the characterization of quasi-Einstein metrics as bases of Einstein warped products implies that no new warped product structures arise from four-dimensional non-reductive homogeneous spaces.

There remain, however, natural directions for future research. One may ask whether similar rigidity persists in higher-dimensional non-reductive homogeneous spaces, or whether new phenomena appear in the quasi-Einstein framework. Another avenue would be to explore related Einstein-type generalizations, such as generalized Ricci solitons, and to test whether comparable equivalences occur. Investigating these questions could further clarify the landscape of rigidity phenomena in geometric analysis and provide deeper insight into the interplay between Einstein-type metrics, warped products, and geometric flows.

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