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Algebraic realization of chain maps in differential graded algebras over a principal ideal domain

M. Benkhalifa

Department of Mathematics, College of Sciences, University of Sharjah, UAE

mbenkhalifa@sharjah.ac.ae

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Abstract: Let R be a principal ideal domain, and let $(T(V),\partial)$ and $(T(W),\delta)$ be two free differential graded R-algebras. Let (V,d) and (W,d') denote the chain complexes of the indecomposables of $(T(V),\partial)$ and $(T(W),\delta)$, respectively. Given a chain map $\xi_*\colon (V,d)\to(W,d')$, this paper addresses the problem of determining whether there exists a DGA-map $\alpha\colon (T(V),\partial)\to(T(W),\delta)$ such that $H_*(\alpha)=H_*(\xi_*)$.

Key words: Free chain algebras, n-characteristic extensions, Coherent morphisms, Adams-Hilton model.

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1. Introduction

Let R be a principal ideal domain (PID), and let $(T(V),\partial)$ and $(T(W),\delta)$ be two free differential graded R-algebras (DGAs for short), with $V_0=W_0=0$. Let (V,d) and (W,d') denote the chain complexes of the indecomposables of $(T(V),\partial)$ and $(T(W),\delta)$, respectively. This paper addresses the following realization problem:

Problem. Find the necessary and sufficient conditions for a given chain map $\xi_*\colon (V,d)\to(W,d')$ to admit an algebraic realization. That is, determine when there exists a DGA-map $\alpha\colon (T(V),\partial)\to(T(W),\delta)$ such that $H_*(\alpha)=H_*(\xi_*)$.

Our approach to this problem is inductive. Assume that a DGA-map $\omega\colon (T(V_{\leq n}),\partial)\to(T(W_{\leq n}),\delta)$ has already been constructed, satisfying $\tilde{\omega}_{\leq n}=\xi_{\leq n}$, where $\tilde{\omega}_{\leq n}\colon (V_{\leq n},d)\to(W_{\leq n},d')$ is the chain map induced by α. Drawing inspiration from the methods outlined in [2, 3, 4, 5], we proceed to extend ω by associating two elements with the DGAs $(T(V),\partial)$ and $(T(W),\delta)$, namely:

$$[\theta_{T(V)}]\in \operatorname{Ext}(H_n(T(V)),\Lambda_n(T(V))),\quad [\theta_{T(W)}]\in \operatorname{Ext}(H_n(T(W)),\Lambda_n(T(W))),$$

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where the R-modules $\Lambda_n(T(V))$ and $\Lambda_n(T(W))$ are given by the relation (2.4). We refer to these as the n-characteristic extensions of $(T(V),\partial)$ and $(T(W),\delta)$, respectively. These extensions support two key concepts introduced in this paper: an *n-coherent morphism* and a *coherent morphism*, both of which involve a chain map $\xi_*\colon (V,d)\to(W,d')$, subject to certain algebraic conditions (see Definitions 3.1 and 4.3).

The main result of this paper is summarized in the following two theorems.

THEOREM 1.1. *Let $(T(V),\partial)$ and $(T(W),\delta)$ be two DGAs. If*

$$\xi_*\colon (V,d)\longrightarrow(W,d')$$

is an n-coherent morphism, there exists a DGA-map

$$\chi\colon (T(V_{\leq n+1}),\partial)\longrightarrow(T(W_{\leq n+1}),\delta)$$

such that $\tilde{\chi}_{\leq n+1}=\xi_{\leq n+1}$, where $\tilde{\chi}_{\leq n+1}\colon (V_{\leq n+1},d)\to(W_{\leq n+1},d')$ is the chain map induced by χ.

THEOREM 1.2. *Let $(T(V),\partial)$ and $(T(W),\delta)$ be two DGAs, and let*

$$\xi_*\colon (V,d)\longrightarrow(W,d')$$

be a chain map. If ξ_* is a coherent morphism, then ξ_* admits an algebraic realization.

As a topological application of our results, we present the following corollary concerning the Adams-Hilton model of a simply connected CW-complex X. Recall that this model is the DGA $(T(V_*(X)),\delta)$, where the free graded R-module $V_*(X)$ satisfies the following relation:

$$V_n(X)=H_{n+1}(X^{n+1},X^n),\qquad \forall n\geq 1.$$

Here $H_{n+1}(X^{n+1},X^n)$ denotes the (cellular) homology module of the pair (X^{n+1},X^n), which can be described as the free R-module generated by the $(n+1)$-cells of X, where X^n is the n-skeleton of X.

COROLLARY 1.3. *Let X,Y be two simply connected CW-complexes. If $\xi_*\colon V_*(X)\to V_*(Y)$ is a coherent isomorphism, then the Adams-Hilton models of X and Y are quasi-isomorphic.*

2. The n-characteristic extensions

Let R be a PID and let $V = (V_i)_{i>0}$ be a graded R-module. The tensor algebra $T(V)$ of V is defined as:

$$T(V) = R \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

This algebra consists of all finite tensor products of elements from V. Let $T_n(V)$ denote the module of the elements of graduation degree n.

A free differential graded algebra is a tensor algebra $T(V)$ equipped with a differential ∂ that satisfies:

- ∂ decreases the degree by 1: $\partial\colon T_n(V) \to T_{n-1}(V)$,
- $\partial^2 = 0$ (the differential is nilpotent),
- The Leibniz rule holds: for $a, b \in T_n(V)$ and $b \in T_m(V)$,

$$\partial(a \otimes b) = \partial(a) \otimes b + (-1)^{\deg(a)} a \otimes \partial(b).$$

The linear component of the differential, $d\colon V_n \to V_{n-1}$, is defined by the relation:

$$\partial(v) - d(v) \in (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

indicating that the non-linear terms of the differential ∂ involve higher-order tensor products. The map d itself forms a differential, and the pair (V, d) is referred to as the chain complex of the indecomposables of $(T(V), \partial)$.

Now let $(T(V), \partial)$ be a DGA. Let us consider the following long sequence:

$$\cdots \to V_{n+1} \xrightarrow{\beta_{n+1}} H_n(T(V_{\leq n})) \xrightarrow{j_n} V_n \xrightarrow{\beta_n} H_{n-1}(T(V_{\leq n-1})) \xrightarrow{j_{n-1}} V_{n-1} \to \cdots$$

in which, for all $n \geq 2$, we have $\operatorname{im} j_n = \ker \beta_n$. The homomorphism j_n is defined by setting:

$$j_n(v_n + y + \operatorname{im} \partial_{\leq n}) = v_n. \tag{2.1}$$

Recall that a homology class in $H_n(T(V_{\leq n}))$ can be represented as $v_n + y + \operatorname{im} \partial_{\leq n}$, where $v_n \in V_n$, $y \in T_n(V_{\leq n-1})$ and $\partial(v_n + y) = 0$. Here $\partial_{\leq n}$ denotes the restriction of the differential ∂ to $T_n(V_{\leq n})$. We define the homomorphism β_{n+1} by:

$$\beta_{n+1}(v_{n+1}) = \partial(v_{n+1}) + \operatorname{im} \partial_{\leq n}. \tag{2.2}$$

Note that the cycle $\partial(v_{n+1})$ needs not to be a boundary in the DGA $T_n(V_{\leq n})$ equipped with the differential $\partial_{\leq n}$.

Recall that the linear part d of the differential ∂, which appears in the chain complex of the indecomposables (V,d), satisfies the relation:

$$d_n = j_{n-1} \circ \beta_n. \tag{2.3}$$

For every $n \geq 2$, we define the R-module $\Lambda_n(T(V))$ by setting:

$$\Lambda_n(T(V)) = \ker j_n. \tag{2.4}$$

The homomorphism j_n, defined in (2.1), gives rise to the short exact sequence:

$$\Lambda_n(T(V)) \rightarrowtail H_n(T(V_{\leq n})) \overset{j_n}{\twoheadrightarrow} \ker \beta_n.$$

As V_n is a free abelian group, $\ker \beta_n \subset V_n$ is also free and the later short exact sequence splits. So we can choose a section $\sigma_n \colon \ker \beta_n \to H_n(T(V_{\leq n}))$ of j_n, i.e.,

$$j_n \circ \sigma_n = id, \tag{2.5}$$

and a splitting:

$$\begin{gathered} \mu_n \colon H_n(T(V_{\leq n})) \overset{\cong}{\longrightarrow} \Lambda_n(T(V)) \oplus \ker \beta_n, \\ \mu_n(x) = (x - \sigma_n \circ j_n(x)) \oplus j_n(x). \end{gathered} \tag{2.6}$$

Remark 2.1. From the formula (2.6), it follows that if $x \in \Lambda_n(T(V))$, then $\mu_n(x) = x$ i.e., the map μ_n is the inclusion $\Lambda_n(T(V)) \subset H_n(T(V_{\leq n}))$.

Consider the differential of the cellular complex $d_{n+1} : V_{n+1} \to V_n$. If $(\operatorname{im} d_{n+1})' \subset V_{n+1}$ denotes a copy isomorphic to the free abelian subgroup $\operatorname{im} d_{n+1} \subset V_n$, then we get the following direct sum:

$$V_{n+1} = \ker d_{n+1} \oplus (\operatorname{im} d_{n+1})'. \tag{2.7}$$

Then notice that

$$(\operatorname{im} d_{n+1})' \overset{d_{n+1}}{\rightarrowtail} \ker d_n \twoheadrightarrow H_n(V,d), \tag{2.8}$$

can be chosen as a free resolution of $H_n(V,d)$, where (V,d) is the chain complex of the indecomposables of $(T(V),\partial)$.

Now, let us consider the homomorphism

$$\mu_n \circ \beta_{n+1} \colon V_{n+1} = \ker d_{n+1} \oplus (\operatorname{im} d_{n+1})' \to H_n(T(V_{\leq n})) \overset{\cong}{\to} \Lambda_n(T(V)) \oplus \ker \beta_n.$$

Due to (2.6), it follows that for every $z + l \in \ker d_{n+1} \oplus (\operatorname{im} d_{n+1})'$, we have

$$\begin{aligned}\mu_n \circ \beta_{n+1}(z+l) &= \Big(\beta_{n+1}(z+l) - \sigma_n \circ j_n \circ \beta_{n+1}(z+l)\Big) \oplus j_n \circ \beta_{n+1}(z+l)\\ &= \Big(\beta_{n+1}(z+l) - \sigma_n \circ d_{n+1}(z+l)\Big) \oplus d_{n+1}(z+l)\\ &= \Big(\beta_{n+1}(z) + \beta_{n+1}(l) - \sigma_n \circ d_{n+1}(l)\Big) \oplus d_{n+1}(l).\end{aligned}$$

Here we use that $z \in \ker d_{n+1}$ and apply the relation (2.3). As a result, for every $z \in \ker d_{n+1}$ we obtain:

$$\mu_n \circ \beta_{n+1}(z) = \beta_{n+1}(z),$$

and since we have:

$$j_n \circ \mu_n \circ \beta_{n+1}(z) = j_n \circ \beta_{n+1}(z) = d_{n+1}(z) = 0,$$

we deduce that:

$$\mu_n \circ \beta_{n+1} = \beta_{n+1} \colon \ker d_{n+1} \longrightarrow \Lambda_n(T(V)) = \ker j_n. \tag{2.9}$$

Likewise, using (2.3) and (2.5), we deduce that:

$$\begin{aligned}j_n \circ (\beta_{n+1}(l) - \sigma_n \circ d_{n+1}(l)) &= j_n \circ \beta_{n+1}(l) - j_n \circ \sigma_n \circ d_{n+1}(l)\\ &= d_{n+1}(l) - d_{n+1}(l) \;=\; 0,\end{aligned}$$

it follows that $\beta_{n+1} - \sigma_n \circ d_{n+1}$ is a homomorphism from $(\operatorname{im} d_{n+1})'$ to $\Lambda_n(T(V))$.

Thus, we define the homomorphism $\theta_{T(V)} \colon (\operatorname{im} d_{n+1})' \to \Lambda_n(T(V))$ by:

$$\theta_{T(V)} = \beta_{n+1} - \sigma_n \circ d_{n+1}. \tag{2.10}$$

Hence, taking into account the resolution (2.8), we obtain the extension class:

$$[\theta_{T(V)}] \in \operatorname{Ext}(H_n(V,d), \Lambda_n(T(V))).$$

Definition 2.2. We refer to the class $[\theta_{T(V)}]$ as the characteristic extension of $(T(V), \partial)$.

Remark 2.3. The following facts are well-known:

1. Let $\omega\colon (T(V),\partial) \to (T(W),\delta)$ a DGA-map. This induces a chain map $\tilde{\omega}\colon (V,d) \to (W,d')$, on the modules of the indecomposables. In turn, this chain map induces a graded homomorphism:

$$H_*(\tilde{\omega})\colon H_*(V,d) \longrightarrow H_*(W,d').$$

   Additionally, this induces the following homomorphism:

$$(H_n(\tilde{\omega}))^*\colon \mathrm{Ext}(H_n(W),\Lambda_n(T(W))) \longrightarrow \mathrm{Ext}(H_n(V),\Lambda_n(T(W))). \tag{2.11}$$

2. Moore's Theorem [6] asserts that ω is a quasi-isomorphism if and only the chain map $\tilde{\omega}$ is a quasi-isomorphism.
3. Furthermore, the DGA-map ω induces a homomorphism:

$$\Lambda(\omega)\colon \Lambda(T(V)) \longrightarrow \Lambda(T(W)), \tag{2.12}$$

   where $\Lambda(\omega)$ is the restriction of homomorphism:

$$H_n(\omega)\colon H_n(T(V_{\leq n})) \longrightarrow H_n(T(W_{\leq n})).$$

   Here the DGA-map $\omega\colon T(V_{\leq n}) \to T(W_{\leq n})$ is the restriction of ω.
4. The DGA-map $\omega\colon T(V_{\leq n}) \to T(W_{\leq n})$ implies the following commutative diagram:

$$\begin{array}{ccc} H_n(T(V_{\leq n})) & \xrightarrow{\ j_n\ } & V_n \\ \big\downarrow{\scriptstyle H_n(\omega)} & & \big\downarrow{\scriptstyle \tilde{\omega}_n} \\ H_n(T(W_{\leq n})) & \xrightarrow{\ j'_n\ } & W_n \end{array} \tag{2.13}$$

   where j_n (respect. j'_n) is defined in (2.1).
5. Using the splitting μ_n, given in (2.6), we obtain the following diagram:

$$\begin{array}{ccc} H_n(T(V_{\leq n})) & \xrightarrow{\ \mu_n\ } & \Lambda_n(T(V)) \oplus \ker\beta_n \\ \big\downarrow{\scriptstyle H_n(\omega)} & & \big\downarrow{\scriptstyle \Lambda_n(\omega)\oplus\tilde{\omega}_n} \\ H_n(T(W_{\leq n})) & \xrightarrow{\ \mu'_n\ } & \Lambda_n(T(W)) \oplus \ker\beta'_n \end{array}$$

where $\sigma'_n\colon \ker\beta'_n \to H_n(T(W_{\leq n}))$ is a section of the homomorphism of $j'_n\colon H_n(T(W_{\leq n})) \to W_n$ (see (2.8)). Taking into account the commutativity of the diagram (2.13), It is easy to check that:

$$(\Lambda_n(\omega)\oplus\tilde{\omega}_n)\circ\mu_n - \mu'_n\circ H_n(\omega) = H_n(\omega)\circ\sigma_n\circ j_n - \sigma'_n\circ j'_n\circ H_n(\omega). \quad (2.14)$$

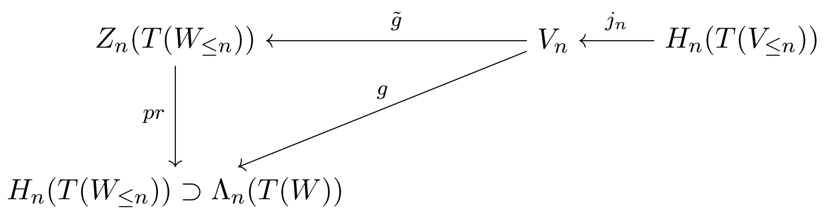
The following lemma will be needed subsequently.

Lemma 2.4. *Let $(T(V),\partial)$ and $(T(W),\delta)$ be two DGAs and let*

$$\omega\colon T(V_{\leq n}) \longrightarrow T(W_{\leq n})$$

be a DGA-map. If $g \in \operatorname{Hom}\big(V_n, \Lambda_n(T(W))\big)$, then there exists a DGA-map $\eta\colon T(V_{\leq n}) \to T(W_{\leq n})$ such that $H_n(\eta) = H_n(\omega) + g\circ j_n$. Moreover, $\tilde{\eta}_* = \tilde{\omega}_*\colon (V_{\leq n}, d) \to (W_{\leq n}, d')$.

Proof. First, let $Z_n(T(W_{\leq n}))$ be the R-submodule of n-cycles of $T(W_{\leq n})$. Since V_n is a free abelian group, there exists a homomorphism $\tilde{g}$ making the following diagram commute:



(2.15)

Next, the homomorphism $\tilde{g}$ allows us to define a map $\eta\colon T(V_{\leq n}) \to T(W_{\leq n})$, by setting:

$$\eta(v) = \begin{cases} \omega(v) + \tilde{g}(v) & \text{if } v \in V, \\ \omega(v) & \text{if } v \in V_{\leq n-1}. \end{cases} \quad (2.16)$$

The map η is a DGA-map. To see this and for $v \in V_n$, we compute:

$$\delta\circ\eta(v) = \delta(\omega(v)) + \delta(\tilde{g}(v)) = \delta\circ\omega(v) = \omega\circ\partial(v) = \eta\circ\partial(v).$$

Here we use the fact that $\tilde{g}(v) \in Z_n(T(W_{\leq n}))$, which indicates that it is indeed a cycle. Additionally, η and ω coincide on $V_{\leq n-1}$. Observe that $\partial(v)$

lies $T(V_{\leq n-1})$. Now, let $x+\operatorname{im}\partial \in H_n(T(V_{\leq n}))$, write $x = v+y$, where $v \in V_n$ and $y \in T(V_{\leq n-1})$. Using the relations (2.16), we get:

$$\begin{aligned} H_n(\eta)(v + y + \operatorname{im}\partial) &= \eta(v + y) + \operatorname{im}\delta \;=\; \eta(v) + \eta(y) + \operatorname{im}\delta \\ &= \omega(v) + \tilde{g}(v) + \omega(y) + \operatorname{im}\delta \\ &= (\omega(v + y) + \operatorname{im}\delta) + (\tilde{g}(v) + \operatorname{im}\delta), \end{aligned} \tag{2.17}$$

and by virtue of the formula (2.1), we can write $\tilde{g}(v) = \tilde{g} \circ j_n(v + y + \operatorname{im}\partial)$. Consequently, taking into consideration the commutativity of the diagram (2.15), the relation (2.17) becomes:

$$\begin{aligned} H_n(\eta)(v + y + \operatorname{im}\partial) &= (\omega(v + y) + \operatorname{im}\delta) + \tilde{g} \circ j_n(v + y + \operatorname{im}\partial) + \operatorname{im}\delta \\ &= H_n(\omega)(v + y) + \operatorname{im}\partial) + pr \circ \tilde{g} \circ j_n(v + y + \operatorname{im}\partial) \\ &= H_n(\omega)(v + y + \operatorname{im}\partial) + g \circ j_n(v + y + \operatorname{im}\partial), \end{aligned}$$

as desired. Finally, since $pr \circ \tilde{g} = g$ and $\operatorname{im} g \subset \Lambda_n(T(W))$, it follows that $\tilde{g}(v)$ is a decomposable cycle in $Z_n(T(W_{\leq n}))$. Therefore, the chain map $\tilde{\eta}_*$ induced by the DGA-map η_* on the decomposables satisfies $\tilde{\eta}_* = \tilde{\omega}_*$. ∎

3. The n-coherent morphisms

With the necessary groundwork in place, we are now ready to introduce the concept of n-coherent morphisms. This notion extends classical conditions, ensuring extending DGA-maps at higher homological levels and offering a more refined understanding of maps between DGAs. With this framework established, we can now proceed to formally define n-coherent morphisms.

Definition 3.1. Let $(T(V), \partial)$ and $(T(W), \delta)$ be two DGAs. We say that a chain map $\xi_*\colon (V, d) \to (W, d')$ is an n-coherent morphism if there exists a DGA-map $\omega\colon (T(V_{\leq n}), \partial) \to (T(W_{\leq n}), \delta)$ such that the following conditions are satisfied:

1. $\xi_{\leq n} = \tilde{\omega}_{\leq n}$, where $\tilde{\omega}_{\leq n}\colon (V_{\leq n}, d) \to (W_{\leq n}, d')$ is the chain map induced by ω on the indecomposables.
2. The following two diagrams commute:

$$\begin{array}{ccc} \ker d_{n+1} & \xrightarrow{\xi_{n+1}} & \ker d'_{n+1} \\ \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta'_{n+1}} \\ \Lambda_n(T(V)) & \xrightarrow{\Lambda_n(\omega)} & \Lambda_n(T(W)) \end{array} \qquad \begin{array}{ccc} \ker \beta_{n+1} & \xrightarrow{\tilde{\omega}} & \ker \beta'_{n+1} \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma'_n} \\ H_n(T(V_{\leq n})) & \xrightarrow{H_n(\omega)} & H_n(T(W_{\leq n})) \end{array} \tag{3.1}$$

where $\Lambda_n(\omega)$ is the homomorphism given in (2.12) and where σ_n and σ'_n are the sections given in (2.5).

3. If $[\theta_{T(V)}]$ (respect. $[\theta_{T(W)}]$) is the n-characteristic extension of $(T(V), \partial)$ (respect. of $(T(W), \delta)$), then we have:

$$(H_n(\tilde{\omega}))^*([\theta_{T(W)}]) = (\Lambda_n(\omega))_*([\theta_{T(V)}]), \tag{3.2}$$

where the homomorphisms $(H_n(\tilde{\omega}))^*$ and $(\Lambda_n(\omega))_*$ are given in (2.11) and (2.12) respectively.

Remark 3.2. Since $\xi_*\colon (V, d) \to (W, d')$ is a chain map, it follows that $\xi_{n+1}(\ker d_{n+1}) \subset \ker d'_{n+1}$. Moreover, the condition (1) implies that the following diagram commutes:

$$\begin{array}{ccc} V_{n+1} & \xrightarrow{\quad \xi_{n+1} \quad} & W_{n+1} \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow{\scriptstyle d'_{n+1}} \\ V_n & \xrightarrow{\quad \xi_n = \tilde{\omega}_n \quad} & W_n \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle d'_n} \\ V_{n-1} & \xrightarrow{\quad \xi_{n-1} = \tilde{\omega}_{n-1} \quad} & W_{n-1} \end{array} \tag{3.3}$$

which leads us to define the homomorphism $H_n(\tilde{\omega})\colon H_n(V) \to H_n(W)$.

Remark 3.3. It is essential to highlight that the formula (3.2) means the following. From (2.8), we know that $(\operatorname{im} d_{n+1})' \overset{d_{n+1}}{\rightarrowtail} \ker d_n \twoheadrightarrow H_n(V)$ is a free resolution of $H_n(V)$. Moreover, to the extensions $[\theta_{T(V)}]$, $[\theta_{T(W)}]$ and to the homomorphisms $(H_n(\tilde{\omega}))^*$, $(\Lambda_n(\omega))_*$ correspond the following two diagrams:

$$\begin{array}{l} (\operatorname{im} d_{n+1})' \overset{d_{n+1}}{\rightarrowtail} \ker d_n \twoheadrightarrow H_n(V) \\ \quad\downarrow{\scriptstyle \theta_{T(V)}} \\ \Lambda_n(T(V)) \\ \quad\downarrow{\scriptstyle \Lambda_n(\omega)} \\ \Lambda_n(T(W)) \end{array} \qquad \begin{array}{lcr} (\operatorname{im} d_{n+1})' \overset{d_{n+1}}{\rightarrowtail} \ker d_n & \twoheadrightarrow & H_n(V) \\ \quad\downarrow{\scriptstyle \xi_{n+1}} & & \downarrow{\scriptstyle H_n(\tilde{\omega})} \\ (\operatorname{im} d'_{n+1})' \overset{d_{n+1}}{\rightarrowtail} \ker d'_n & \twoheadrightarrow & H_n(W) \\ \quad\downarrow{\scriptstyle \theta_{T(W)}} & & \\ \Lambda_n(T(W)) & & \end{array}$$

where $(H_n(\tilde{\omega}))^*([\theta_{T(W)}]) = [\theta_{T(W)} \circ \xi_{n+1}]$ and $(\Lambda_n(\omega))_*([\theta_{T(V)}]) = [\Lambda_n(\omega) \circ \theta_{T(V)}]$ which implies that:

$$(H_n(\tilde{\omega}))^*([\theta_{T(W)}]) - (\Lambda_n(\omega))_*([\theta_{T(V)}]) = [\theta_{T(W)} \circ \xi_{n+1} - \Lambda_n(\omega) \circ \theta_{T(V)}].$$

Hence, the relation (3.2) is equivalent to the existence of a homomorphism g,

$$\ker d_n \xrightarrow{g} \Lambda_n(T(W)),$$

satisfying the relation:

$$\theta_{T(W)} \circ \xi_{n+1} - \Lambda_n(\omega) \circ \theta_{T(V)} = g \circ d_{n+1}. \tag{3.4}$$

Note that, since $V_n = \ker d_n \oplus (\operatorname{im} d_n)'$ (see (2.7)), we can extend the homomorphism $g\colon \ker d_n \to \Lambda_n(T(W))$ to a homomorphism (also denoted by g):

$$g\colon V_n \longrightarrow \Lambda_n(T(W)), \tag{3.5}$$

by requiring that g is zero on $(\operatorname{im} d_n)'$.

Example 3.4. If $\alpha\colon (T(V), \partial) \to (T(W), \delta)$ is a DGA-map, then the chain map $\tilde{\alpha}_* : (V, d) \to (W, d')$, induced by α on the indecomposables (see Remark 2.3) is an n-coherent morphism for every $n \geq 1$, as the conditions of Definition 3.1 are clearly satisfied in this case.

Example 3.5. Let $(T(V), \partial)$ and $(T(W), \delta)$ be two DGAs such that the graded module $H_n(V)$ and $H_n(W)$ are R-free. In this case the condition 3 in Definition 3.1 is trivially satisfied, as we have:

$$\operatorname{Ext}(H_n(V), \Lambda_n(T(V))) = \operatorname{Ext}(H_n(W), \Lambda_n(T(W))) = 0.$$

4. Main result

The following lemma is essential for establishing the main result of the paper.

Lemma 4.1. *Let $(T(V), \partial)$ and $(T(W), \delta)$ be two DGAs and let*

$$\omega\colon T(V_{\leq n}) \longrightarrow T(W_{\leq n})$$

be a DGA-map. If $\rho\colon V_{n+1} \to W_{n+1}$ is a homomorphism making the following diagram commute:

$$\begin{array}{ccc} V_{n+1} & \xrightarrow{\quad\rho\quad} & W_{n+1} \\ {\scriptstyle\beta_{k+1}}\big\downarrow & & \big\downarrow{\scriptstyle\beta'_{k+1}} \\ H_n(T(V_{\leq n})) & \xrightarrow{\;H_n(\omega)\;} & H_n(T(W_{\leq n})). \end{array} \tag{4.1}$$

Then ω can be extended to a DGA-map $\chi\colon T(V_{\leq n+1}) \to T(W_{\leq n+1})$ such that the homomorphism $\tilde{\chi}_{n+1}\colon V_{n+1} \to W_{n+1}$, induced by the DGA-map χ on V_{n+1}, satisfies $\tilde{\chi}_{n+1} = \rho$.

Proof. For every $v \in V_{n+1}$, using the formula (2.2), we get:

$$H_n(\omega) \circ \beta_{n+1}(v) - \beta'_{n+1} \circ \rho(v) = \omega \circ \partial(v) - \delta \circ \rho(v) + \operatorname{im} \delta_{\leq n}.$$

Since the diagram (4.1) commutes, the element $\omega \circ \partial(v) - \delta \circ \rho(v) \in \operatorname{im} \delta_{\leq n}$, therefore there exists an element $y_v \in T_{n+1}(W_{\leq n})$ such that:

$$\omega \circ \partial(v) - \delta \circ \rho(v) = \delta(y_v). \tag{4.2}$$

Thus, we define $\chi\colon (T(V_{\leq n+1}), \partial) \to (T(W_{\leq n+1}), \delta)$ by setting:

$$\chi(v) = \begin{cases} \rho(v) + y_v & \text{if } v \in V_{n+1}, \\ \omega(v) & \text{if } v \in V_{\leq n}. \end{cases}$$

By (4.2), we have:

$$\delta \circ \chi(v) = \delta \circ \rho(v) + \delta(y_v) = \beta_n \circ \varphi_n(v_\xi) = \omega \circ \partial(v) = \chi \circ \partial(v).$$

Here we use that $\partial(v) \in T_n(W_{\leq n})$ and $\chi = \omega$ for every $v \in V_{\leq n}$. As a result, χ is a DGA-map. Finally, since the element $y_v \in T_{n+1}(W_{\leq n})$, it is clear that χ satisfies $\tilde{\chi}_{n+1} = \rho$. ∎

We are now ready to present the main result of this paper.

THEOREM 4.2. *Let $(T(V), \partial)$ and $(T(W), \delta)$ be two DGAs. If*

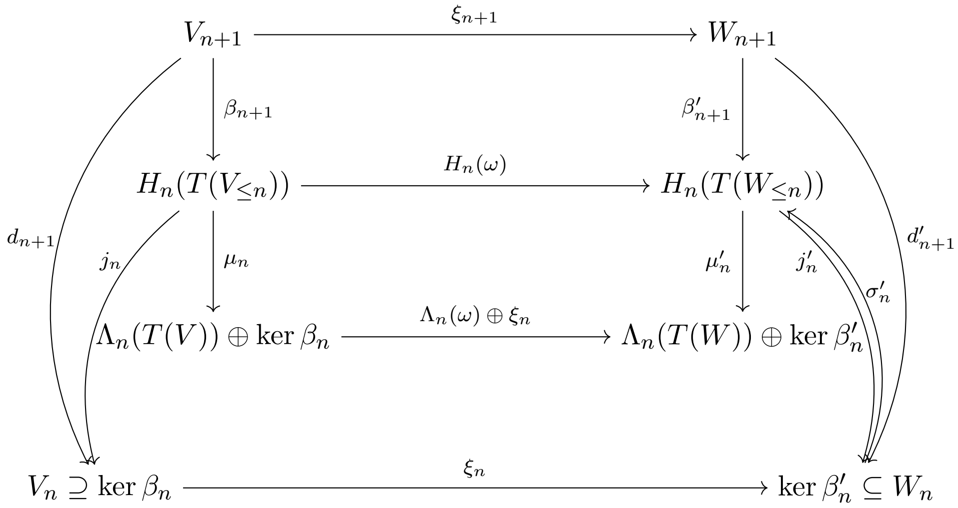
$$\xi_*\colon (V, d) \longrightarrow (W, d')$$

is an n-coherent morphism, there exists a DGA-map

$$\chi\colon (T(V_{\leq n+1}), \partial) \longrightarrow (T(W_{\leq n+1}), \delta)$$

such that $\tilde{\chi}_{\leq n+1} = \xi_{\leq n+1}$.

Proof. First, since $\xi_*\colon (V,d) \to (W,d')$ is an n-coherent morphism, there exists a DGA-map $\omega\colon (T(V_{\leq n}),\partial) \to (T(W_{\leq n}),\delta)$ such that the conditions of Definition 3.1 are satisfied. Next, let us consider the following diagram:



First, by (2.9) we have $\beta_{n+1}(\ker d_{n+1}) \subset \Lambda_n(T(V))$. Thus, by applying diagram (3.1), for every $z \in \ker d_{n+1}$, we deduce that:

$$\begin{aligned} &H_n(\omega)\circ\beta_{n+1}(z) - \beta'_{n+1}\circ\xi_{n+1}(z) \\ &\qquad = \Lambda_n(\omega)\circ\beta_{n+1}(z) - \beta'_{n+1}\circ\xi_{n+1}(z) \;=\; 0. \end{aligned} \tag{4.3}$$

Next, on one hand, using (2.14), we get:

$$\begin{aligned} &(\Lambda_n(\omega)\oplus\xi_n)\circ\mu_n\circ\beta_{n+1} - \mu'_n\circ H_n(\omega)\circ\beta_{n+1} \\ &\qquad = H_n(\omega)\circ\sigma_n\circ j_n\circ\beta_{n+1} - \sigma'_n\circ j'_n\circ H_n(\omega)\circ\beta_{n+1}, \end{aligned}$$

and taking into account the formula (2.3) and the diagram (2.13), it follows that:

$$\begin{aligned} &(\Lambda_n(\omega)\oplus\xi_n)\circ\mu_n\circ\beta_{n+1} - \mu'_n\circ H_n(\omega)\circ\beta_{n+1} \\ &\qquad = H_n(\omega)\circ\sigma_n\circ d_{n+1} - \sigma'_n\circ\tilde{\omega}_n\circ j_n\circ\beta_{n+1} \\ &\qquad = H_n(\omega)\circ\sigma_n\circ d_{n+1} - \sigma'_n\circ\tilde{\omega}_n\circ d_{n+1} \\ &\qquad = (H_n(\omega)\circ\sigma_n - \sigma'_n\circ\tilde{\omega}_n)\circ d_{n+1}. \end{aligned}$$

But the diagram (3.1) implies that $H_n(\omega)\circ\sigma_n - \sigma'_n\circ\tilde{\omega}_n = 0$. As a result, we get:

$$(\Lambda_n(\omega)\oplus\xi_n)\circ\mu_n\circ\beta_{n+1} = \mu'_n\circ H_n(\omega)\circ\beta_{n+1}. \tag{4.4}$$

On the other hand, using (2.6) and (2.10), we get:

$$(\Lambda_n(\omega) \oplus \xi_n) \circ \mu_n \circ \beta_{n+1} = \Lambda_n(\omega) \circ (\beta_{n+1} - \sigma_n \circ d_{n+1}) \ = \ \Lambda_n(\omega) \circ \theta_{T(V)},$$
$$\mu'_n \circ \beta'_{n+1} \circ \xi_{n+1} = (\beta'_{n+1} - \sigma'_n \circ d'_{n+1}) \circ \xi_{n+1} \ = \ \theta_{T(W)} \circ \xi_{n+1},$$

and combining with the formula (4.4), we deduce that:

$$\mu'_n \circ (\beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1}) = \theta_{T(W)} \circ \xi_{n+1} - \Lambda_n(\omega) \circ \theta_{T(V)}.$$

Next, using (3.4) we obtain:

$$\mu'_n \circ (\beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1}) = g \circ d_{n+1}, \tag{4.5}$$

where the homomorphism $g\colon V_n \to \Lambda_n(T(W))$ is given in (3.5).

By virtue of (2.13) and the formula (1) in Definition 3.1, we get:

$$\begin{aligned} j'_n \circ (\beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1}) &= j'_n \circ \beta'_{n+1} \circ \xi_{n+1} - j'_n \circ H_n(\omega) \circ \beta_{n+1} \\ &= j'_n \circ \beta'_{n+1} \circ \xi_{n+1} - \tilde{\omega}_n \circ j_n \circ \beta_{n+1} \\ &= d'_{n+1} \circ \xi_{n+1} - \xi_n \circ d_{n+1} \ = \ 0, \end{aligned}$$

it follows that $\operatorname{im}(\beta'_{n+1} \circ \xi_{n+1} - \pi_n(\omega) \circ \beta_{n+1}) \subset \Lambda_n(T(W))$ and applying Remark 2.1, we obtain:

$$\mu'_n \circ (\beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1}) = \beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1},$$

therefore, the relation (4.5) becomes:

$$\beta'_{n+1} \circ \xi_{n+1} - H_n(\omega) \circ \beta_{n+1} = g \circ d_{n+1}, \tag{4.6}$$

in the other words, using again (2.3), it follows that:

$$\beta'_{n+1} \circ \xi_{n+1} - (H_n(\omega) + g \circ j_n) \circ \beta_{n+1} \ = \ 0. \tag{4.7}$$

Now, according to Lemma 2.4, the map $\omega\colon (T(V_{\leq n}), \partial) \to (T(W_{\leq n}), \delta)$ and the homomorphism g allow us to define a map $\eta\colon (T(V_{\leq n}), \partial) \to (T(W_{\leq n}), \delta)$ such that:

$$H_n(\eta) = H_n(\omega) + g \circ j_n \qquad \text{and} \qquad \tilde{\eta}_* = \tilde{\omega}_*. \tag{4.8}$$

Here we invoke the relation (3.5) extending g. Consequently, the following diagram commute:

$$\begin{array}{ccc} V_{n+1} & \xrightarrow{\xi_{n+1}} & W_{n+1} \\ \Big\downarrow{\scriptstyle \beta_{n+1}} & & \Big\downarrow{\scriptstyle \beta'_{n+1}} \\ H_n(T(V_{\leq n})) & \xrightarrow{H_n(\eta)} & H_n(T(W_{\leq n})). \end{array}$$

Indeed, due to (4.7) and (4.8), an easy computation shows that:

$$H_n(\eta)\circ\beta_{n+1} = (H_n(\omega) + g\circ j_n)\circ\beta_{n+1} = \beta'_{n+1}\circ\xi_{n+1}.$$

Hence, applying Lemma 4.1, we can extend η to get a DGA-map:

$$\chi\colon (T(V_{\leq n+1}),\partial) \longrightarrow (T(W_{\leq n+1}),\delta),$$

such that $\tilde{\chi}_{n+1} = \xi_{n+1}$. As χ is an extension of the map η to $T(V_{\leq n+1})$ and taking into account (4.8), it follows that:

$$\tilde{\chi}_{\leq n} = \tilde{\omega}_{\leq n}\colon T(V_{\leq n}) \longrightarrow T(W_{\leq n}),$$

as we wanted. ∎

Definition 4.3. Let $(T(V),\partial)$ and $(T(W),\delta)$ be two DGAs. We say that a chain map $\xi_*\colon (V,d)\to(W,d')$ is a coherent morphism if ξ_* is an n-coherent morphism for every $n\geq 1$.

Theorem 4.4. *Let $(T(V),\partial)$ and $(T(W),\delta)$ be two DGAs, and let*

$$\xi_*\colon (V,d)\longrightarrow (W,d')$$

be a chain map. If ξ_* is a coherent morphism, then ξ_* admits an algebraic realization.

Proof. First, for a fixed n, as ξ_* is an n-coherent morphism, there exists a DGA-map $\omega\colon (T(V_{\leq n}),\partial)\to(T(W_{\leq n}),\delta)$ satisfying the conditions of Definition 3.1. Applying Theorem 4.2, we obtain a DGA-map $\chi\colon (T(V_{\leq n+1}),\partial)\to (T(W_{\leq n+1}),\delta)$ such that $\tilde{\chi}_{\leq n+1} = \xi_{\leq n+1}$. Next, repeating this proceeds for every n, we can construct a DGA-map $\alpha\colon (T(V),\partial)\to(T(W),\delta)$ satisfying $\tilde{\alpha}_* = \xi_*$. Consequently, we get $H_*(\alpha) = H_*(\xi_*)$. Thus, ξ_* admit an algebraic realization. ∎

Corollary 4.5. *Let* $(T(V), \partial)$ *and* $(T(W), \delta)$ *be two DGAs and let*

$$\xi_* \colon (V, d) \longrightarrow (W, d')$$

be a quasi-isomorphism. If ξ_* *is a coherent morphism, then* $(T(V), \partial)$ *and* $(T(W), \delta)$ *are quasi-isomorphic.*

Proof. By virtue of Theorem 4.4, ξ_* admit an algebraic realization. That means a DGA-map $\alpha \colon (T(V), \partial) \to (T(W), \delta)$ satisfying $H_*(\alpha) = H_*(\xi_*)$. Given that ξ_* is a quasi-isomorphism, it implies that the induced homomorphism on homology, $H_*(\xi_*)$ is an isomorphism. As a result, the DGA-map α must also be a quasi-isomorphism. This conclusion follows from Moore's Theorem, as referenced in Remark 2.3. ∎

5. Topological applications

Recall that the Adams-Hilton model [1] of a simply connected CW-complex X is a quasi-isomorphism of algebras:

$$(T(V_*(X)), \delta_X) \xrightarrow{\simeq} C_*(\Omega X), \tag{5.1}$$

where $C_*(\Omega X)$ is the singular chain complex of the loop space of X and where the free graded R-module $V_*(X)$ satisfies the following relation:

$$V_n(X) = H_{n+1}(X^{n+1}, X^n) \qquad \forall n \geq 1.$$

Here $H_{n+1}(X^{n+1}, X^n)$ denotes the (cellular) homology module of the pair (X^{n+1}, X^n), which can be described as the free module generated by the $(n+1)$-cells of X, where X^n is the k-skeleton of X. Recall that for a cell e in the cellular complex of X, if ∂e denotes its boundary in the cellular chain complex of X, then the differential δ_X on the generator e is defined by:

$$\delta_X(e) = -(\partial e).$$

From the quasi-isomorphism (5.1), it follows that:

$$H_*(T(V_*(X)), \delta_X) \cong H_*(\Omega X, R), \qquad H_*(V_*(X), d_X) \cong H_*(X, R).$$

Here, $(V_*(X), d)$ represents the chain complex of indecomposables of the DGA $(T(V_*(X)), \delta_X)$, which can be identified with the cellular chain complex associated with the CW-complex X.

Corollary 5.1. *Let X and Y be two simply connected CW-complexes and let $\xi_*\colon (V_*(X), d_X) \to (V_*(Y), d_Y)$ is chain map. If ξ_* is a coherent morphism, then there exists a DGA-map $\alpha\colon (T(V_*(X)), \delta_X) \to (T(V_*(Y)), \delta_Y)$ such that $H_*(\alpha) = H_*(\xi_*)$. Moreover, if ξ_* is a quasi-isomorphism, then the Adams-Hilton models $(T(V_*(X)), \delta_X)$ and $(T(V_*(Y)), \delta_Y)$ of X and Y respectively, are also quasi-isomorphic.*

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