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Fourth Hankel and Toeplitz determinant estimates for certain analytic functions associated with Four Leaf function

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Abstract: The objective of this paper is to establish initial coefficient inequalities, Upper bounds to the Hankel and Toeplitz determinants for certain normalized univalent functions defined on the open unit disk $\mathbb{D}$ in the complex plane related to the analytic function $\varphi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ that maps the open unit disk in the complex plane onto the interior of four leaf shaped domain in the right half of the complex plane.

Key words: Univalent functions; Hankel determinants; Starlike functions; Coefficient inequalities; Four Leaf domain; Toeplitz determinants.

MSC (2020): 30C45, 30C80.

1. Introduction

Let $\mathcal{A}$ be the family of analytic functions f defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. The Taylor series expansion of $f \in \mathcal{A}$ is

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{for all } z \in \mathbb{D}, \text{ where } a_n = \frac{f^{(n)}(0)}{n!}. \tag{1.1}$$

The collection of univalent functions (that are one-to-one) $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. Unless otherwise stated throughout this paper, we assume the series representation of $f \in \mathcal{S}$ is of the form (1.1).

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The classes of starlike, convex and bounded turning functions denoted by $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ respectively are well recognised subclasses of $\mathcal{S}$. Let $\mathcal{B}$ be the family of analytic functions $w(z)$ in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. The members of $\mathcal{B}$ are called the Schwarz functions and $w(z) = z,\ w(z) = z^2$ are a couple of examples of members in $\mathcal{B}$. A function $f \in \mathcal{A}$ is said to be subordinate to $g \in \mathcal{A}$ if there exists a $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In this case, we write $f \prec g$. If g is univalent, then $f \prec g$ if, and only if, $g(0) = f(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For basic information on univalent function theory, we refer to [7] and [24].

For $f \in \mathcal{A}$ with series expansion (1.1),The q^{th} Hankel determinant of index n, denoted by $H_{q,n}(f)$ (or simply $H_q(n)$), is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

for $q \geq 2$ and $n \geq 1$ with $a_1 = 1$ (see [23], [22]), whereas q^{th} Symmetric Toeplitz determinant of index n, denoted by $T_q(n)$, is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

for $q \geq 2$ and $n \geq 1$ with $a_1 = 1$ (see [33]).

The classes S^* and $\mathcal{C}$ were further generalized to the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ by Ma and Minda [18] and studied growth, distortion results associated with these classes, which are defined as

$$\mathcal{S}^*(\varphi) = \Big\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \Big\},$$
$$\mathcal{C}(\varphi) = \Big\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \Big\},$$

where $\varphi \in \mathcal{A}$ such that $\Re\{\varphi(z)\} > 0$, $\varphi'(0) > 0$, $\varphi(\mathbb{D})$ is symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$. If we choose $\varphi(z) = \frac{1+z}{1-z}$ for $z \in \mathbb{D}$ then it is evident that $\mathcal{S}^*(\varphi) = \mathcal{S}^*$ and $\mathcal{C}(\varphi) = \mathcal{C}$.

Zalcman conjecture: In 1960, Lawrence Zalcman conjectured that the coefficients of $f \in \mathcal{S}$ with series representation (1.1) satisfy the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \qquad \text{for } n \geq 2.$$

Ma [17] proposed generalized Zalcman conjecture for $f \in \mathcal{S}$ of the form (1.1) that

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \qquad \text{for } n, m \geq 2$$

and proved this conjecture is true for starlike functions and univalent function with real coefficients.

2. Literature review

Geometric function theory relies heavily on the study of Hankel and Toeplitz determinants related to the members of $\mathcal{S}$.Many researchers were inspired to study $|H_2(2)|$ and $|H_3(1)|$ for different subclasses of $\mathcal{S}$ by the groundbreaking work of Pommerenke [22],[23] and Hayman [9] on Hankel determinants for functions in the class $\mathcal{S}$. For instance, Noonan and Thomas [20] studied the second Hankel determinant of areally mean p-valent functions, Noor [21] has investigated the Hankel determinant of close-to-convex univalent functions. Babalola [4] estimated an upper bound of $|H_{3,1}(f)|$ for the functions in the classes $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$. Sharp estimates of $|H_{3,1}(f)|$ for these three classes were obtained in the papers [15], [14] and [13] respectively.

In recent years, various geometric characteristics have been investigated and considered by selecting a certain function φ that satisfies the conditions proposed by Ma-Minda. For instance, the functions $\varphi_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, $\varphi_C(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, $\varphi_S(z) = 1 + \sin z$ maps $\mathbb{D}$ onto three-leaf, cardioid, and eight-shaped domains respectively (see [5], [28], and [3]) are a few notable Ma-Minda-type functions that have been researched recently. The classes

$$\mathcal{S}^*_{4L} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi_{4L}(z) \right\},$$
$$\mathcal{C}_{4L} = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_{4L}(z) \right\},$$
$$\mathcal{R}_{4L} = \left\{ f \in \mathcal{S} : f'(z) \prec \varphi_{4L}(z) \right\}$$

of starlike, convex and bounded turning functions are associated with

$\varphi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ that maps $\mathbb{D}$ onto four leaf shaped domain have been introduced by Pongsakorn Sunthrayuth *et al.* [32].

2.1. Literature Review of Concerning Upper bound of $|H_{4,1}(f)|$ for $f \in \mathcal{S}$: Arif *et al.* [2] estimated an upper bound of $|H_{4,1}(f)|$ for $f \in \mathcal{R}$ by proving $|H_4(1)| \leq \frac{73757}{94500}$. Later, an upper bound for $|H_{4,1}(f)|$ for $f \in \mathcal{S}$ related to different geometric domains was obtained by few Âáresearchers. We now list some of them.

An upper bound for $|H_{4,1}(f)|$ is obtained:

(1) for the class $\mathcal{R}_1 = \{f \in \mathcal{A} : \Re\{f'(z) + zf''(z)\} > 0\}$ by Kaur *et al.* [10];

(2) for the class $\mathcal{R}(\varphi_C) = \{f \in \mathcal{A} : f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2\}$ by Srivastava *et al.* [30];

(3) for the class $f \in \mathcal{R}_{\sin} = \{f \in \mathcal{A} : f'(z) \prec 1 + \sin z\}$ by Khan *et al.* [11];

(4) for the class $\mathcal{R}_{\mathrm{SG}} = \{f \in \mathcal{A} : f'(z) \prec \frac{2}{1+e^{-z}} \text{ for all } z \in \mathbb{D}\}$ by Khan *et al.* [12];

(5) for the class $\mathcal{R}_1(\cos z) = \{f \in \mathcal{A} : f'(z) + zf''(z) \prec cosz\}$ by Yakaiah and Bharavi Sharma [35];

(6) for the class $\mathcal{R}_1(1 + \sin z) = \{f \in \mathcal{A} : f'(z) + zf''(z) \prec 1 + \sin z\}$ by Ganesh *et al.* [6].

(7) For recent investigations on Hankel determinents,we refer to the research works of H.M. Srivatsava *et al.* [31]. They found upper bounds for the third and fourth order Hankel determinants for the functions of new subclasses of analytic functions by making use of subordination involving the sine function and the modified sigmoid activation function.

2.2. Literature Review on Toeplitz Determinants:

(1) The Hankel and Toeplitz determinants were closely related.Toeplitz determinants contain constant entries along the principal diagonal, unlike Hankel determinants.

(2) Thomas and Abdul Halim [33] initiated the concept of Toeplitz matrices $T_q(n)$, for the functions in $\mathcal{S}^*$ and close to convex functions $\mathcal{K}$.

(3) Zhang *et al.* [36] studied an upper bounds of the fourth Toeplitz determinant for the class $\mathcal{S}^*_s = \left\{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z)\right\}$.

(4) Vijayalakshmi *et al.* [34] studied symmetric Toeplitz determinants for classes defined by post quantum operators subordinated to the limacon function.

(5) Srivastava *et al.* [29] studied Hankel and Toeplitz determinants for a subclass of q-Starlike functions associated with a General Conic Domain.

(6) Yakaiah and Bharavi Sharma [35] estimated fourth Toeplitz determinants for $f \in \mathcal{R}_1(\cos z)$

(7) Yakaiah *et al.* [6] computed upper bounds for $|T_4(1)|$ and $|T_4(2)|$ for $f \in \mathcal{R}_1(1+\sin z)$.

(8) Recently, Mandal *et al.* [19] investigated Toeplitz determinants of logarithmic coefficients of inverse functions for certain classes of univalent function.

(9) For similar type of studies concerning Toeplitz determinants for starlike, convex and bounded boundary rotation functions we refer to [1] and [25].

3. Motivation and identification of research problem

(1) Gunasekar *et al.* [8] studied a new subclass $\mathcal{A}_4^{r,s}$ of analytic functions related to the four-leaf domain,where $r \geq 0, s \in [0,1]$ and

$$\mathcal{A}_4^{r,s} = \left\{ f \,:\, (1-r)(1-s)\frac{f(z)}{z} + (s+r(1+s))f'(z) + rs(zf''(z)-2) \prec \varphi_{4L}(z) \right\}.$$

(2) Shaba *et al.* [27] studied Fekete-Szegö problem and second Hankel determinant for a subclass of bi-univalent functions associated with four leaf domain.

Motivated by the works of Sunthrayuth *et al.*, Gunasekar *et al.*, Shaba *et al.* and Yakaiah *et al.* [32], [8], [27], [35], in this paper, upper bounds of fourth Hankel and Toeplitz determinants for the class

$$\mathcal{R}_1\big(\varphi_{4L}\big) = \left\{ f \in \mathcal{S} \,:\, f'(z) + zf''(z) \prec \varphi_{4L}(z) \right\}$$

associated with four leaf function $\varphi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ were computed. The image of $\mathbb{D}$ under $\varphi_{4L}(z)$ can be seen as in Figure 1. The organization of this paper is as follows. In Section 4, we state some lemmas to prove our main results. The initial coefficient bounds, second order Hankel and Toeplitz determinants bound estimates for functions in the class $\mathcal{R}_1\big(\varphi_{4L}\big)$ are presented in Section 5 and the third order Hankel and Toeplitz determinants bound estimates in Section 6. Finally, Section 7 is dedicated to compute upper bound estimates of fourth order Hankel and Toeplitz determinants for $f \in \mathcal{R}_1(\varphi_{4L})$.

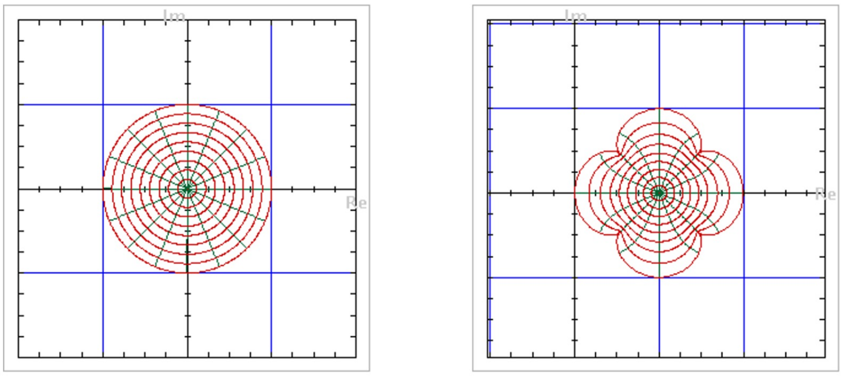

Figure 1: The Image of unit disk under the mapping φ_{4L}
(Using Complex Tools)

4. A set of useful lemmas

The collection of analytic functions $p(z)$ defined on the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ is called the class of functions with positive real part, and it is denoted by $\mathcal{P}$. For $p \in \mathcal{P}$, we have the following series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad \text{for } z \in \mathbb{D}. \tag{4.1}$$

Unless otherwise stated throughout this paper, we assume the series representation of $p \in \mathcal{P}$ is of the form (4.1).

Lemma 4.1. ([24]) *If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for any positive integer n. The inequality is sharp for $p(z) = \dfrac{1+z}{1-z}$.*

Lemma 4.2. ([18]) *If $p \in \mathcal{P}$ and $\rho \in \mathbb{C}$, then $|c_2 - \rho c_1^2| \leq 2\max\{1, |2\rho - 1|\}$. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \dfrac{1+z^2}{1-z^2}$.*

LEMMA 4.3. ([3]) *If $p \in \mathcal{P}$, then for any real numbers A, B and C,*

$$\left|Ac_1^3 - Bc_1c_2 + Cc_3\right| \le 2\left(|A| + |B - 2A| + |A - B + C|\right).$$

LEMMA 4.4. ([26]) *If $p \in \mathcal{P}$, then for all $n, m \in \mathbb{N}$,*

$$|\rho c_n c_m - c_{n+m}| = \begin{cases} 2 & \text{if } 0 \le \rho \le 1, \\ 2|2\rho - 1| & \text{otherwise.} \end{cases}$$

This inequality is sharp.

LEMMA 4.5. ([26]) *If $p \in \mathcal{P}$, l, m, n and r be real numbers and the inequalities $0 < m < 1$, $0 < r < 1$,*

$$\begin{aligned} 8r(1-r)\Big((mn - 2l)^2 + (m(r+m) - n)^2\Big) + m(1-m)(n - 2rm)^2 \\ \le 4m^2(1-m)^2 r(1-r) \end{aligned} \tag{4.2}$$

hold, then

$$\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right| \le 2.$$

LEMMA 4.6. ([16]) *If $p \in \mathcal{P}$ and $c_1 \ge 0$, then*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2), \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

5. Initial coefficient inequalities, second Hankel determinants for the class $\mathcal{R}_1(\varphi_{4L})$

Let $f \in \mathcal{R}_1(\varphi_{4L})$. Then there exists $w \in \mathcal{B}$ such that

$$f'(z) + zf''(z) = \varphi_{4L}(w(z)), \qquad \text{for all } z \in \mathbb{D}. \tag{5.1}$$

If we take

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \in \mathcal{P} \qquad \text{for all } z \in \mathbb{D} \text{ then } w(z) = \frac{p(z) - 1}{p(z) + 1}$$

so that

$$f'(z) + zf''(z) = 1 + \frac{5}{6}\Big(\frac{p(z)-1}{p(z)+1}\Big) + \frac{1}{6}\Big(\frac{p(z)-1}{p(z)+1}\Big)^5 \quad \text{for all } z \in \mathbb{D}. \tag{5.2}$$

On substituting (1.1) and (4.1) in (5.2) and comparing like coefficients on both the sides of (5.2), we obtain

$$a_2 = \frac{5}{48}c_1, \tag{5.3}$$

$$a_3 = \frac{5}{108}\Big(c_2 - \frac{1}{2}c_1^2\Big), \tag{5.4}$$

$$a_4 = \frac{5}{192}\Big(\frac{1}{4}c_1^3 - c_1c_2 + c_3\Big), \tag{5.5}$$

$$a_5 = \frac{1}{60}\Big(c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4\Big), \tag{5.6}$$

$$a_6 = \frac{5}{432}\Big(c_5 - c_1c_4 - c_2c_3 + \frac{3}{4}c_1^2c_3 + \frac{3}{4}c_1c_2^2 - \frac{1}{2}c_1^3c_2 + \frac{3}{40}c_1^5\Big), \tag{5.7}$$

$$a_7 = \frac{5}{588}\Big(c_6 - c_1c_5 - c_2c_4 - \frac{1}{2}c_3^2 + \frac{3}{4}c_1^2c_4 + \frac{3}{2}c_1c_2c_3 + \frac{1}{4}c_2^3 - \frac{1}{2}c_1^3c_3 - \frac{3}{4}c_1^2c_2^2 + \frac{3}{8}c_1^4c_2 - \frac{1}{16}c_1^6\Big). \tag{5.8}$$

Example 5.1. By taking the Schwarz functions $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$ and $w(z) = z^4$ in (5.1) followed by integrating on both sides and utilizing the fact $f(0) = 0$, $f'(0) = 1$ we get respectively:

(1) $f_1(z) = z + \frac{5}{24}z^2 + \frac{1}{216}z^6$,

(2) $f_2(z) = z + \frac{5}{54}z^3 + \frac{1}{726}z^{11}$,

(3) $f_3(z) = z + \frac{5}{96}z^4 + \frac{1}{1536}z^{16}$,

(4) $f_4(z) = z + \frac{1}{30}z^5 + \frac{1}{2646}z^{21}$.

It is easy to see that $f_i \in \mathcal{R}_1(\varphi_{4L})$ for $i = 1, 2, 3, 4$.

We now estimate initial coefficient bounds for the functions in $\mathcal{R}_1(\varphi_{4L})$.

Theorem 5.1. *If $f \in \mathcal{R}_1(\varphi_{4L})$ is given by (1.1). Then $|a_2| \le \frac{5}{24}$, $|a_3| \le \frac{5}{54}$, $|a_4| \le \frac{5}{96}$, $|a_5| \le \frac{1}{30}$, $|a_6| \le \frac{5}{72}$ and $|a_7| \le \frac{235}{1764}$. The members f_1, f_2, f_3, f_4 are extremal functions for first four inequalities respectively.*

Proof. Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). By applying Lemma 4.1, Lemma 4.2 and Lemma 4.3 to (5.3), (5.4) and (5.5) respectively, we obtain

$$\begin{aligned}
|a_2| &= \left|\frac{5}{48}c_1\right| \leq \frac{5}{24},\\
|a_3| &= \left|\frac{5}{108}\left(c_2 - \frac{c_1^2}{8}\right)\right| \leq \frac{5}{54},\\
|a_4| &= \frac{5}{768}\left|c_1^3 - 4c_1c_2 + 4c_3\right| \leq \frac{5}{768}[2(1+2+1)] = \frac{5}{96},\\
|a_5| &= \frac{1}{60}\left|c_4 - c_1c_3 - \frac{1}{2}c_2^2 + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4\right|\\
&= \frac{1}{60}\left|\frac{1}{8}c_1^4 + \frac{1}{2}c_2^2 + c_1c_3 - \frac{3}{4}c_1^2c_2 - c_4\right|\\
&= \frac{1}{60}\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right|,
\end{aligned}$$

where $l = 1/8$, $r = 1/2$, $m = 1/2$ and $n = 1/2$. These values of l, m, n and r satisfy the inequality (4.2) in the hypothesis of Lemma 4.5 as it is evident that

$$8r(1-r)\Big((mn-2l)^2 + (m(r+m)-n)^2\Big) + m(1-m)(n-2rm)^2 = 0,$$
$$4m^2(1-m)^2r(1-r) = 0.0625$$

as well as $0 < m < 1$, $0 < r < 1$. Therefore, by Lemma 4.5,

$$\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right| \leq 2$$

and hence $|a_5| \leq \frac{1}{30}$.

By Lemma 4.1, Lemma 4.4 and Lemma 4.5, we have $|c_1| \leq 2$, $\left|c_5 - c_2c_3\right| \leq 2$ and $\left|\frac{3}{40}c_1^4 + \frac{3}{4}c_2^2 + \frac{3}{4}c_1c_3 - \frac{1}{2}c_1^2c_2 - c_4\right| \leq 2$.

Consequently, a simple computation shows that

$$\begin{aligned}
|a_6| &= \frac{5}{432}\left|c_5 - c_1c_4 - c_2c_3 + \frac{3}{4}c_1^2c_3 + \frac{3}{4}c_1c_2^2 - \frac{1}{2}c_1^3c_2 + \frac{3}{40}c_1^5\right|,\\
&\leq \frac{5}{432}\left(\left|c_5 - c_2c_3\right| + |c_1|\left|\frac{3}{40}c_1^4 + \frac{3}{4}c_2^2 + \frac{3}{4}c_1c_3 - \frac{1}{2}c_1^2c_2 - c_4\right|\right)\\
&\leq \frac{5}{432}(2+4) = \frac{5}{72}.
\end{aligned}$$

Further, in view of Lemma 4.3, we have $\left|-\frac{3}{8}c_1^3+c_1c_2-\frac{1}{2}c_3\right|\leq\frac{3}{2}$ and in view of Lemma 4.4, we have $|c_6-c_1c_5|\leq 2$, $|c_4-\frac{3}{4}c_1c_3|\leq 2$, $|c_3-\frac{3}{2}c_1c_2|\leq 4$, $|c_2-\frac{3}{2}c_1^2|\leq 4$ and using the fact $|c_n|\leq 2$ for $n\geq 1$ and in view of Lemma4.5, we obtain

$$\begin{aligned}|a_7| &= \frac{5}{588}\Big(c_6-c_1c_5-c_2c_4-\frac{1}{2}c_3^2+\frac{3}{4}c_1^2c_4+\frac{3}{2}c_1c_2c_3+\frac{1}{4}c_2^3\\ &\qquad -\frac{1}{2}c_1^3c_3-\frac{3}{4}c_1^2c_2^2+\frac{3}{8}c_1^4c_2-\frac{1}{16}c_1^6\Big)\\ &\leq \frac{5}{588}\left(\frac{3}{8}|c_1^2|\left|\frac{1}{12}c_1^4+\frac{1}{2}c_2^2+\frac{2}{3}c_1c_3-\frac{1}{2}c_1^2c_2-c_4\right|+\frac{1}{2}|c_6-c_1c_5|\right.\\ &\qquad \left.+\frac{1}{2}|c_2|\left|c_4-\frac{3}{4}c_1c_3\right|+\frac{1}{4}|c_3|\left|c_3-\frac{3}{2}c_1c_2\right|+\frac{1}{8}|c_2|^2\left|c_2-\frac{3}{2}c_1^2\right|\right)\\ &\leq \frac{5}{588}\Big(3+\frac{2}{6}+\frac{4}{2}+2+2\Big) = \frac{235}{1764}.\end{aligned}$$

∎

We now obtain an upper bound for Fekete-Szegö functional of the class $\mathcal{R}_1(\varphi_{4L})$.

THEOREM 5.2. *If* $f\in\mathcal{R}_1(\varphi_{4L})$ *is given by* (1.1), *then for any* $\rho\in\mathbb{C}$, *we have*

$$|a_3-\rho a_2^2|\leq\frac{5}{54}\max\left\{1,\frac{15}{32}|\rho|\right\} \tag{5.9}$$

and this inequality is sharp.

Proof. Let $f\in\mathcal{R}_1(\varphi_{4L})$ and $\rho\in\mathbb{C}$. Then in view of Lemma 4.2, we obtain

$$\begin{aligned}|a_3-\rho a_2^2| &= \left|\frac{5}{108}\left(c_2-\frac{1}{2}c_1^2\right)-\rho\left(\frac{5}{48}\right)^2c_1^2\right|\\ &= \frac{5}{108}\left|c_2-\left(\frac{32+15\rho}{64}\right)c_1^2\right|\\ &\leq \frac{5}{54}\max\left\{1,\left|2\left(\frac{32+15\rho}{64}\right)-1\right|\right\} = \frac{5}{54}\max\left\{1,\frac{15}{32}|\rho|\right\}.\end{aligned}$$

SHARPNESS: Case (i): If $|\rho|\leq\frac{32}{15}$ then $|a_3-\rho a_2^2|\leq\frac{5}{54}$ and the function $f_2(z)=z+\frac{5}{54}z^3+\frac{1}{726}z^{11}\in\mathcal{R}_1(\varphi_{4L})$ is an extremal function for this inequality.

Case (ii): If $|\rho| \geq \frac{32}{15}$ then $|a_3 - \rho a_2^2| \leq \frac{25}{576}|\rho|$ and the function $f_1(z) = z + \frac{5}{24}z^2 + \frac{1}{216}z^6 \in \mathcal{R}_1(\varphi_{4L})$ is an extremal function for this inequality. ∎

We now estimate an upper bound for second Hankel determinants for the class $\mathcal{R}_1(\varphi_{4L})$.

THEOREM 5.3. *If $f \in \mathcal{R}_1(\varphi_{4L})$, then $|H_{2,2}(f)| \leq \dfrac{25}{2916}$ and this inequality is sharp.*

Proof. Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). A simple computation by using (5.3), (5.4) and (5.5) and applying the Lemma 4.6, we obtain

$$\begin{aligned}
|H_{2,2}(f)| &= |a_2a_4 - a_3^2| \\
&= \frac{25}{144}\left|\frac{c_1}{256}\left(c_1^3 - 4c_1c_2 + 4c_3\right) - \frac{1}{81}\left(c_2^2 + \frac{1}{4}c_1^4 - c_1^2c_2\right)\right| \\
&= \frac{25}{11664}\left|\frac{17}{256}c_1^4 - \frac{17}{64}c_1^2c_2 + \frac{81}{64}c_1c_3 - c_2^2\right| \\
&= \frac{25}{11664}\left|\frac{17}{256}c_1^4 - \frac{17}{64}\frac{c_1^2}{2}\left(c_1^2 + x(4-c_1^2)\right)\right. \\
&\quad + \frac{81}{64}\frac{c_1}{4}\left(c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\right) \\
&\quad \left. - \frac{1}{4}\left(c_1^2 + x(4-c_1^2)\right)^2\right| \\
&\leq \frac{25}{11664}\left(\frac{81}{256}c^2(4-c^2)|x|^2 + \frac{1}{4}(4-c^2)^2|x|^2\right. \\
&\quad \left. + \frac{81}{128}c(4-c^2)(1-|x|^2)c^2|\delta|\right) \\
&\leq \frac{25}{11664}\left(\frac{81}{256}c^2(4-c^2)t^2 + \frac{1}{4}(4-c^2)^2t^2 + \frac{81}{128}c(4-c^2)(1-t^2)c^2\right)
\end{aligned}$$

where $|\delta| \leq 1$. Let

$$F(c,t) = \frac{81}{256}c^2(4-c^2)t^2 + \frac{1}{4}(4-c^2)^2t^2 + \frac{81}{128}c(4-c^2)(1-t^2)c^2$$

where $c = |c_1| \in [0,2]$, $t = |x| \in [0,1]$ and $0 \leq z \leq 1$. It is easy to see that

$$\frac{\partial F}{\partial t} = (4-c^2)2t\left(\frac{81}{256}c^2 + \frac{1}{4}(4-c^2) - \frac{81}{128}c\right) \geq 0.$$

Therefore, $F(c,t)$ is an increasing function in variable t. Consequently

$$\begin{aligned}|F(c,t)| \le F(c,1) &= \frac{81}{256}c^2\big(4-c^2\big)+\frac{1}{4}\big(4-c^2\big)^2+\frac{81}{128}c\big(4-c^2\big)(0)\\ &= \frac{81}{256}c^2\big(4-c^2\big)+\frac{1}{4}\big(4-c^2\big)^2 = \psi(c).\end{aligned}$$

Now $\psi'(c)=0$ implies $c=0$ or c is a root of $\frac{81}{64}(2-c^2)+(c^2-4)=0$. Since $\psi''(0)=-\frac{47}{32}<0$ at $c=0$. Therefore, ψ attains its maximum at $c=0$. Hence

$$|H_{2,2}(f)| \le \frac{25}{11664}\psi(0)=\frac{25}{11664}(4)=\frac{25}{2916}.$$

SHARPNESS: Consider $f_2(z)=z+\frac{5}{54}z^3+\frac{1}{726}z^{11}\in\mathcal{R}_1\big(\varphi_{4L}\big)$ as in the Example 5.1. It is easy to see that $|H_{2,2}(f_2)|=|a_2a_4-a_3^2|=\left|0-\big(\frac{5}{54}\big)^2\right| = \frac{25}{2916}$. ∎

THEOREM 5.4. *If* $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$, *then* $|H_{2,3}(f)|\le\dfrac{1109}{82944}$.

Proof. Let $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$ be given by (1.1). A simple computation by using (5.4), (5.5) and (5.6), we obtain

$$\begin{aligned}|H_{2,3}(f)| &= \frac{1}{144}\bigg|\frac{1}{9}c_2c_4+\frac{97}{1152}c_1c_2c_3+\frac{31}{2304}c_1^2c_2^2-\frac{31}{4608}c_1^4c_2\\ &\quad -\frac{1}{18}c_2^3-\frac{1}{18}c_1^2c_4+\frac{31}{4608}c_1^3c_3+\frac{31}{36864}c_1^6-\frac{25}{256}c_3^2\bigg|\\ &\le \frac{1}{144}\bigg(\frac{1}{18}|c_1|^2\bigg|\frac{31}{2048}c_1^4+\frac{31}{128}c_2^2+\frac{31}{256}c_1c_3-\frac{31}{256}c_1^2c_2-c_4\bigg|\\ &\quad +\frac{1}{9}|c_2|\bigg|c_4-\frac{1}{2}c_2^2\bigg|+\frac{25}{256}|c_3|\bigg|c_3-\frac{194}{225}c_1c_2\bigg|+\frac{31}{768}|c_1|^2|c_2|^2\bigg).\end{aligned}$$

An application of Lemma 4.5 shows that

$$\left|\frac{31}{2048}c_1^4+\frac{31}{128}c_2^2+\frac{31}{256}c_1c_3-\frac{31}{256}c_1^2c_2-c_4\right|\le 2.$$

Further, it is easy to see that an application of Lemma 4.4 yields $\left|c_3-\frac{194}{225}c_1c_2\right| \le 2$,and Lemma 4.2 yields $\left|c_4-\frac{1}{2}c_2^2\right|\le 2$. In view of these inequalities and using the fact that $|c_n|\le 2$ for all $n\ge 1$, we finally obtain $|H_{2,3}(f)|\le\frac{1109}{82944}\approxeq 0.01337$. ∎

6. Bounds of Zalcman functionals, third Hankel and Toeplitz determinants for the class $\mathcal{R}_1(\varphi_{4L})$

Theorem 6.1. *If $f \in \mathcal{R}_1(\varphi_{4L})$, then $|a_4 - a_2 a_3| \leq \frac{5}{96}$. The inequality is sharp for $f_3(z) = z + \dfrac{5}{96} z^4 + \dfrac{1}{1536} z^{16} \in \mathcal{R}_1(\varphi_{4L})$.*

Proof. Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). By using Lemma 4.3, we obtain

$$\begin{aligned} |a_4 - a_2 a_3| &= \frac{5}{20736} \left|37 c_1^3 - 128 c_1 c_2 + 108 c_3\right| \\ &\leq \frac{5}{20736} \left(2\left(37 + 54 + 17\right)\right) = \frac{5}{96}. \end{aligned}$$

∎

Theorem 6.2. *Let $f \in \mathcal{R}_1(\varphi_{4L})$, then*

$$\begin{aligned} \left|a_3 - a_2^2\right| &\leq \frac{5}{54}, \\ \left|a_5 - a_3^2\right| &\leq \frac{1}{30}, \\ \left|a_7 - a_4^2\right| &\leq \frac{61385}{451584}. \end{aligned}$$

Further, $f_2(z) = z + \frac{5}{54} z^3 + \frac{1}{726} z^{11}$, $f_4(z) = z + \frac{1}{30} z^5 + \frac{1}{2646} z^{21}$ as given in Example 5.1 are extremal functions for first two inequalities respectively.

Proof. Taking $\rho = 1$ in the (5.9) yield $\left|a_3 - a_2^2\right| \leq \frac{5}{54}$. Using (5.6) and (5.4), we obtain

$$\begin{aligned} \left|a_5 - a_3^2\right| &= \left|\frac{1}{60}\Big(c_4 - c_1 c_3 - \frac{1}{2} c_2^2 + \frac{3}{4} c_1^2 c_2 - \frac{1}{8} c_1^4\Big) - \frac{25}{11664}\Big(c_2 - \frac{1}{2} c_1^2\Big)^2\right| \\ &= \frac{1}{60}\left|\frac{1833}{11664} c_1^4 + \frac{3666}{5832} c_2^2 + c_1 c_3 - \frac{10248}{11664} c_1^2 c_2 - c_4\right| \\ &= \frac{1}{60}\left|\frac{611}{3888} c_1^4 + \frac{611}{972} c_2^2 + c_1 c_3 - \frac{427}{486} c_1^2 c_2 - c_4\right| \\ &= \frac{1}{60}\left|l c_1^4 + r c_2^2 + 2m c_1 c_3 - \frac{3n}{2} c_1^2 c_2 - c_4\right|, \end{aligned}$$

where $l = 611/3888$, $r = 611/9722$, $m = 1/2$, $n = 427/729$. These values of l, r, m, n satisfy the hypothesis of Lemma 4.5 as it is evident from the facts that

$0<r<1,\ 0<m<1,\ 4m^2(1-m)^2r(1-r)=0.0583655$ and

$$8r(1-r)\Big((mn-2l)^2+\big(m(r+m)-n\big)^2\Big)+m(1-m)(n-2rm)^2=0.00217541.$$

Therefore, by Lemma 4.5,

$$\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3n}{2}c_1^2c_2-c_4\right|\leq 2.$$

Hence $|a_5-a_3^2|\leq\frac{1}{60}(2)=\frac{1}{30}$.

On similar lines, utilizing the inequalities $|a_7|\leq\frac{235}{1764}$ and $|a_4|\leq\frac{5}{96}$ as proved in Theorem 5.1, we obtain $|a_7-a_4^2|\leq|a_7|+|a_4|^2\leq\frac{61385}{451584}$. ∎

Remark 6.1. It is clear from Theorem 6.2 that the Zalcman conjecture is true for $n=2,3,4$ for $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$.

We now estimate an upper bound of $|H_{3,1}(f)|$ for $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$.

THEOREM 6.3. *If* $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$, *then* $|H_{3,1}(f)\leq\dfrac{481}{82944}$.

Proof. Let $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$ be given by (1.1). Then in view of Theorem 6.1 and Theorem 6.2, we have $|a_4-a_2a_3|\leq\frac{5}{96}$, $|a_3-a_2^2|\leq\frac{5}{54}$ and $|a_5-a_3^2|\leq\frac{1}{30}$. On expanding and utilizing the triangular inequality, we have

$$|H_{3,1}(f)|\leq\left|a_5-a_3^2\right|\left|a_3-a_2^2\right|+|a_4-a_2a_3|^2.$$

Hence, $|H_{3,1}(f)|\leq\left(\frac{1}{30}\right)\left(\frac{5}{54}\right)+\left(\frac{5}{96}\right)^2=\frac{481}{82944}\approxeq 0.005799$. ∎

THEOREM 6.4. *If* $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$, *then* $|H_{3,2}(f)|=\dfrac{3065155}{1605632}$.

Proof. Let $f\in\mathcal{R}_1\big(\varphi_{4L}\big)$ be given by (1.1). By using (5.3)–(5.6), we obtain

$$\begin{aligned}|a_2a_5-a_3a_4|&=\frac{1}{576}\left|c_1c_4+\frac{7}{36}c_1c_2^2-\frac{47}{72}c_1^2c_3+\frac{11}{48}c_1^3c_2-\frac{11}{288}c_1^5-\frac{25}{36}c_2c_3\right|\\&=\frac{1}{576}\Bigg|c_1\Big(c_4-\frac{47}{72}c_1c_3\Big)+\frac{10}{48}c_1^3\Big(c_2-\frac{11}{60}c_1^2\Big)\\&\qquad+\frac{1}{48}c_2\Big(c_1^3+\frac{28}{3}c_1c_2-\frac{100}{3}c_3\Big)\Bigg|\end{aligned}$$

$$\leq \frac{1}{576}\Bigg(|c_1|\Big|c_4 - \frac{47}{72}c_1c_3\Big| + \frac{10}{48}|c_1|^3\Big|c_2 - \frac{11}{60}c_1^2\Big| + \frac{1}{48}|c_2|\Big|c_1^3 + \frac{28}{3}c_1c_2 - \frac{100}{3}c_3\Big|\Bigg)$$

$$\leq \frac{1}{576}\Big(4 + \frac{10}{3} + \frac{53}{36}\Big) = \frac{317}{20736}.$$

A simple computation by using the fact that $|a_2a_5 - a_3a_4| \leq \frac{317}{20736}$ and applying the Theorem 5.1, Theorem 5.3 and Theorem 5.4, we obtain

$$\begin{aligned}
|H_{3,2}(f)| &= \Big|a_6\big(a_2a_4 - a_3^2\big) - a_5(a_2a_5 - a_3a_4) + a_4\big(a_3a_5 - a_4^2\big)\Big| \\
&\leq |a_6|\big|a_2a_4 - a_3^2\big| + |a_5||a_2a_5 - a_3a_4| + |a_4|\big|a_3a_5 - a_4^2\big| \\
&\leq \frac{125}{209952} + \frac{317}{622080} + \frac{5545}{7962624} \\
&= \frac{3065155}{1605632} \approxeq 1.9090021.
\end{aligned}$$

∎

THEOREM 6.5. *If* $f \in \mathcal{R}_1\big(\varphi_{4L}\big)$, *then* $|T_3(1)| \leq \dfrac{25553}{23328}$.

Proof. From Theorem 5.1 and Theorem 5.2, we have $|a_2| \leq \frac{5}{24}$, $|a_3| \leq \frac{5}{54}$ and $|a_3 - 2a_2^2| \leq \frac{5}{54}$. Therefore it is easy to see that

$$\begin{aligned}
|T_3(1)| &= 1 + 2|a_2|^2 + |a_3|\big|a_3 - 2a_2^2\big| \\
&\leq 1 + \left(\frac{5}{24}\right)^2 + \frac{25}{2916} = \frac{25553}{23328} \approxeq 1.09537894.
\end{aligned}$$

∎

THEOREM 6.6. *If* $f \in \mathcal{R}_1\big(\varphi_{4L}\big)$, *then* $|T_3(2)| \leq \dfrac{70625}{4478976}$.

Proof. By the definition of $T_3(2)$, we have

$$T_3(2) = (a_2 - a_4)\big(a_2^2 - 2a_3^2 + a_2a_4\big) = (a_2 - a_4)\big(T_2(2) + H_{2,2}(f)\big).$$

Since $f \in \mathcal{R}_1(\varphi_{4L})$, we have $|a_2| \leq 5/24$ and $|a_3| \leq 5/54$ and hence

$$|T_2(2)| = |a_2^2 - a_3^2| \leq |a_2|^2 + |a_3|^2 \leq \frac{25}{576} + \frac{25}{9216} = \frac{2425}{46656}.$$

Further, using the bounds obtained in the Theorem 5.1 and Theorem 5.3, it can be shown that

$$\begin{aligned}
|T_3(2)| &= |a_2 - a_4|\big|T_2(2) + H_{2,2}(f)\big| \\
&\leq \big(|a_2| + |a_4|\big)\big(|T_2(2)| + |H_{2,2}(f)|\big) \\
&\leq \Big(\frac{5}{24} + \frac{5}{96}\Big)\Big(\frac{2425}{46656} + \frac{25}{2916}\Big) \\
&= \frac{70625}{4478976} \approxeq 0.0139506.
\end{aligned}$$

∎

7. Upper bounds of fourth Hankel and Toeplitz determinants for functions in the class $\mathcal{R}_1(\varphi_{4L})$

Theorem 7.1. *If $f \in \mathcal{R}_1(\varphi_{4L})$, then $|a_5 - a_2a_4| \leq \frac{1}{30}$. This inequality is sharp.*

Proof. Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). By using (5.6), (5.3) and (5.5),

$$\begin{aligned}
|a_5 - a_2a_4| &= \frac{1}{60}\Big|\frac{509}{3072}c_1^4 + \frac{1}{2}c_2^2 + \frac{893}{768}c_1c_3 - \frac{701}{768}c_1^2c_2 - c_4\Big| \\
&= \frac{1}{60}\Big|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\Big|
\end{aligned}$$

where $l = 509/3072$, $r = 1/2$, $m = 893/1536$ and $n = 701/1152$. The values of $l, r,\ m,\ n$ satisfy the hypothesis of Lemma 4.5 and hence

$$|a_5 - a_2a_4| = \frac{1}{60}\Big|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\Big| \leq \frac{1}{30}.$$

The equality hold in $|a_5 - a_2a_3| \leq \frac{1}{30}$ for

$$f_4(z) = z + \frac{1}{30}z^5 + \frac{1}{2646}z^{21} \in \mathcal{R}_1(\varphi_{4L}).$$

∎

THEOREM 7.2. *If* $f \in \mathcal{R}_1(\varphi_{4L})$, *then* $|H_{4,1}(f)| \leq 0.00148679$.

Proof. It is found in [11, 6] that if $f \in \mathcal{S}$ of the form (1.1), then

$$\begin{aligned} H_{4,1}(f) = a_7 H_{3,1}(f) &- 2a_4 a_6\left(a_2 a_4 - a_3^2\right) - 2a_5 a_6(a_2 a_3 - a_4) \\ &- a_6^2\left(a_3 - a_2^2\right) + a_5^2\left(a_2 a_4 - a_3^2\right) \\ &+ a_5^2\left(a_2 a_4 + 2a_3^2\right) - a_5^3 + a_4^4 - 3a_3 a_4^2 a_5. \end{aligned}$$

Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). Then the required upper bound is estimated as follows

$$\begin{aligned} |H_{4,1}(f)| \leq & |a_7|\,|H_{3,1}| + 2|a_4||a_6|\left|a_2 a_4 - a_3^2\right| \\ & + 2|a_5||a_6||a_2 a_3 - a_4| + |a_6|^2\left|a_3 - a_2^2\right| \\ & + |a_5|^2\left|a_2 a_4 - a_3^2\right| + |a_5|^2\left|a_2 a_4 + 2a_3^2\right| \\ & + |a_5|^3 + |a_4|^4 + 3|a_3||a_4|^2|a_5| \\ \leq & \, 0.00148679. \end{aligned}$$

follows by using the bounds obtained in Theorem 5.1,Theorem 5.3, Theorem 6.1 and Theorem 6.2. ∎

THEOREM 7.3. *If* $f \in \mathcal{R}_1(\varphi_{4L})$, *then* $|T_4(1)| \leq 1.16244413$.

Proof. Let $f \in \mathcal{R}_1(\varphi_{4L})$ be given by (1.1). Then by using the bounds of initial coefficients of $f \in \mathcal{R}_1(\varphi_{4L})$ obtained in Theorem 5.1, we can prove

$$\begin{aligned} |a_3 - a_2 a_4| \leq |a_3| + |a_2||a_4| \leq \frac{5}{54} + \frac{5}{24}\left(\frac{5}{96}\right) &= \frac{715}{6912}, \\ |a_2 - a_2 a_3| \leq |a_2|\big(1 + |a_3|\big) \leq \frac{5}{24}\left(1 + \frac{5}{54}\right) &= \frac{295}{1296}, \\ |1 - a_2^2| \leq 1 + |a_2|^2 \leq 1 + \frac{25}{576} &= \frac{601}{576}. \end{aligned}$$

Also, using the results established in Theorem 6.2, Theorem 6.1 and Theorem 5.3, respectively, we obtain

$$\begin{aligned}|T_4(1)| &= \Big|\big(1-a_2^2\big)^2 - (a_2a_3-a_4)^2 + \big(a_3^2-a_2a_4\big)^2 - (a_2-a_2a_3)^2 \\ &\quad + 2\big(a_2^2-a_3\big)(a_3-a_2a_4)\Big| \\ &\leq \big|1-a_2^2\big|^2 + |a_2a_3-a_4|^2 + \big|a_3^2-a_2a_4\big|^2 + |a_2-a_2a_3|^2 \\ &\quad + 2\big|a_2^2-a_3\big||a_3-a_2a_4| \\ &\leq 1.16244413.\end{aligned}$$

∎

THEOREM 7.4. *If $f \in \mathcal{R}_1(\varphi_{4L})$ is of the form (1.1), then $|T_4(2)| \leq 0.0036043885$.*

Proof. Since $f \in \mathcal{R}_1(\varphi_{4L})$, in view of Theorem 6.6, Theorem 5.4, Theorem 5.3 and Theorem 6.4, we have $|T_2(2)| \leq \frac{2425}{46656}$, $|H_{2,3}(f)| \leq \frac{3373}{360000}$, $|H_{2,2}(f)| \leq \frac{16}{2025}$ and $|a_3a_4 - a_2a_5| \leq \frac{317}{20736}$. Further,

$$\begin{aligned}|a_2a_3 - a_3a_4| &\leq |a_3|\big(|a_2| + |a_4|\big) \leq \frac{125}{5184}, \\ |a_2a_4 - a_3a_5| &\leq |a_2||a_4| + |a_3||a_5| \leq \frac{25}{2304} + \frac{5}{1620} = \frac{289}{20736},\end{aligned}$$

in view of Theorem 5.1. Therefore,we have

$$\begin{aligned}|T_4(2)| &\leq |T_2(2)|^2 + |a_3a_4 - a_2a_5|^2 + |H_{2,3}(f)|^2 + |a_2a_3 - a_3a_4|^2 \\ &\quad + 2|H_{2,2}(f)||a_2a_4 - a_3a_5| \\ &\leq 0.0036043885.\end{aligned}$$

∎

8. CONCLUDING REMARKS AND SCOPE OF FURTHER RESEARCH

In this paper, upper bounds of Hankel and Toeplitz determinants $|H_{4,1}(f)|$, $|T_4(1)|$ and $|T_4(2)|$ for $f \in \mathcal{R}_1(\varphi_{4L})$ associated with four leaf function $\varphi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ were computed. Examples have been provided to illustrate the sharpness of certain results. There is ample scope to study upper bounds of fourth order Hankel and Toeplitz determinants for functions in various other subclasses of class $\mathcal{S}$ related to four leaf function.

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