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Analytic infinite gaps

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Abstract: We provide infinite-dimensional versions of analytic gap dichotomies, in the sense that a sequence of analytic hereditary families $\{\mathcal{I}_p\}_{p<\omega}$ of subsets of a countable set Ω is either countably separated or there is a tree structure inside Ω in which p-chains are sets from $\mathcal{I}_p$. A topological version of this is that if K is a separable Rosenthal compact space, then either K is a continuous image of a finite-to-one preimage of a metric compactum or there is a tree structure inside K in which p-chains inside every branch form a relatively discrete family of sets.

Key words: Analytic gaps, Rosenthal compacta, infinite-dimensional dichotomies.

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1. Introduction

The classical notion of a gap, as first considered by Hausdorff [11] in set theory can be thought as a couple of families $\mathcal{I}_0$ and $\mathcal{I}_1$ of sets that are *orthogonal* but not *separated*. Here, orthogonal means that $a_0 \cap a_1$ is finite whenever $a_0 \in \mathcal{I}_0$ and $a_1 \in \mathcal{I}_1$. And the families would be separated if there existed disjoint sets b_0, b_1 such that, for $i = 0, 1$ we have $a \subseteq^* b_i$ whenever $a \in \mathcal{I}_i$. Remember that $x \subseteq^* y$ means that $x \setminus y$ is finite. There were two new ingredients that were added in the article [15] (see also [18]) to the theory of gaps. One ingredient is that the assumption of descriptive complexity hypotheses (like being analytic, which is natural in certain applications) translates into strong structural results. A second ingredient was the introduction of a weaker notion of separation: $\mathcal{I}_0$ and $\mathcal{I}_1$ are countably separated if there exists a countable family $\mathcal{F}$ of sets such that for every $a_0 \in \mathcal{I}_0$ and $a_1 \in \mathcal{I}_1$ there exists disjoint $b_0, b_1 \in \mathcal{F}$ such that $a_0 \subseteq b_0$ and $a_1 \subseteq b_1$. One of the main results of [15] was the following dichotomy:

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THEOREM 1.1. ([15, THEOREM 2]) *Let $\mathcal{I}_0$ and $\mathcal{I}_1$ two hereditary analytic families of subsets of ω. One and only one of the following two statements hold:*

1. *either $\mathcal{I}_0$ and $\mathcal{I}_1$ are countably separated,*
2. *or there exists an injective mapping from the dyadic tree $2^{<\omega}$ into ω that sends every 0-chain into an element of $\mathcal{I}_0$ and every 1-chain into an element of $\mathcal{I}_1$.*

TERMINOLOGY: $\mathcal{I}_i$ is hereditary if $b \in \mathcal{I}_i$ and $a \subset b$ implies $a \in \mathcal{I}_i$. We consider a topology on the power set $\mathcal{P}(\omega)$ of ω given by the transfer of the product topology of $\{0,1\}^\omega$ via the natural identification of subsets of ω with sequences of zeros and ones. The property of being analytic refers to this topology. Given $p \in \omega$, a p-chain is a set x of finite sequences of natural numbers such that whenever $s = (s_0, \ldots, s_n)$ and $t = (t_0, \ldots, t_m)$ are different elements of x, then either $n < m$ and t is of the form $t = (s_0, \ldots, s_n, p, t_{n+2}, \ldots, t_m)$ or, vice-versa $m < n$ and $s = (t_0, \ldots, t_m, p, s_{m+2}, \ldots, s_n)$. Another way to view this is to consider, for every $x \in \omega^\omega$ and every $p \in \omega$ the set

$$a_p^x = \{(x_0, \ldots, x_n) \, : \, x_{n+1} = p\}.$$

A p-chain is then a subset of a_p^x for some x.

A further step in this direction of research was taken in [5, 6] by considering finitely many families instead of just two. The families $\mathcal{I}_0, \ldots, \mathcal{I}_{m-1}$ are said to be countably separated if there is a countable family $\mathcal{F}$ such that for every $a_0 \in \mathcal{I}_0, \ldots, a_{m-1} \in \mathcal{I}_{m-1}$ there exists $b_0, \ldots, b_{m-1} \in \mathcal{F}$ such that $\bigcap_{i<m} b_i = \emptyset$ and $a_i \subseteq b_i$ for all i. The n-adic tree $n^{<\omega}$ is the set of all finite sequences of numbers $0, 1, \ldots, n-1$. The new version of the dichotomy was

THEOREM 1.2. ([6]) *Let $\{\mathcal{I}_i : i < n\}$ be n analytic hereditary families of subsets of ω. One and only one of the following two statements hold:*

1. *either the families are countably separated,*
2. *or there exists an injective mapping from the n-adic tree $n^{<\omega}$ into ω that sends every i-chain into an element of $\mathcal{I}_i$.*

The aim of this note is to consider infinite versions of this dichotomy, dealing with countably many families instead of finitely many. We shall state two different versions of it. In one of them the tree involved will be $\omega^{<\omega}$ the set of all finite sequences of natural numbers, while in the other one we will use the profusing tree

$$\Upsilon = \{(t_0, \ldots, t_k) \in \omega^{<\omega} : t_i \leq i \text{ for all } i\}.$$

Accordingly, there will be two infinite generalizations of the concept of countable separation. They are given below for a general family $\mathbb{I}$ of sequences of sets, but keep in mind the case when $\mathbb{I} = \prod_{i<\omega} \mathcal{I}_i$ for hereditary families of sets $\mathcal{I}_0, \mathcal{I}_1, \ldots$. We fix a countable set Ω.

Definition 1.3. Let $\mathbb{I} \subset \mathcal{P}(\Omega)^\omega$ be a family of sequences of subsets of Ω.

1. $\mathbb{I}$ is ϵ-countably separated if there exists a countable family $\mathcal{F} \subset \mathcal{P}(\Omega)^\omega$ such that:
   (a) $\bigcap_{p<\omega} b_p$ is finite whenever $(b_p)_{p<\omega} \in \mathcal{F}$;
   (b) for any given sequence of sets $(a_p) \in \mathbb{I}$ there exists $(b_p) \in \mathcal{F}$ such that $a_p \subseteq b_p$ for all p.
2. $\mathbb{I}$ is E-countably separated if there exists a countable family $\mathcal{G} \subset \mathcal{P}(\Omega)$ such that: for any given sequence of sets $(a_p) \in \mathbb{I}$ there exist n and $b_0, \ldots, b_n \in \mathcal{G}$ such that $\bigcap_{i\leq n} b_i$ is finite and $a_i \subseteq b_i$ for $i = 1, \ldots, n$.

Observe that every E-countably separated $\mathbb{I}$ is ϵ-countably separated, just taking $\mathcal{F}$ the family of all sequences of the form $(b_0, \ldots, b_n, \Omega, \Omega, \Omega, \Omega, \ldots)$ with $b_1 \cap \cdots \cap b_n$ finite and $b_0, \ldots, b_n \in \mathcal{G}$. The converse is not true (cf. Proposition 3.2). We will say that $\mathbb{I}$ is hereditary if whenever $(b_p) \in \mathbb{I}$ and $a_p \subseteq b_p$ for all p, then $(a_p) \in \mathbb{I}$. These are the main results of this paper:

Theorem 1.4. *Let $\mathbb{I} \subset \mathcal{P}(\Omega)^\omega$ be an analytic hereditary family of sequences of subsets of Ω. One and only one of the two following options holds:*

1. *either there exists an injective mapping $\sigma : \omega^{<\omega} \to \Omega$ such that $(\sigma(a_p^x))_{p<\omega} \in \mathbb{I}$ for all $x \in \omega^\omega$,*
2. *or $\mathbb{I}$ is ϵ-countably separated.*

Theorem 1.5. *Let $\mathbb{I} \subset \mathcal{P}(\Omega)^\omega$ be an analytic hereditary family of sequences of subsets of Ω. One and only one of the two following options holds:*

1. *either there is an injective mapping $\sigma : \Upsilon \to \Omega$ such that $(\sigma(a_p^x))_{p<\omega} \in \mathbb{I}$ for all $x \in \omega^\omega$ with $x_i \leq i$ for all i,*
2. *or $\mathbb{I}$ is E-countably separated.*

In both cases, when $\mathbb{I} = \prod_{i<\omega} \mathcal{I}_i$, condition (1) can be rephrased saying that σ sends p-chains into sets from $\mathcal{I}_p$ for every p.

It may sound natural to study the infinite version of separation instead of countable separation. However, the natural notion of separation of infinitely many families seems to be just stating that a finite subfamily is separated. This does not add anything really new. The notions of countable separation studied here cannot be reduced to the finite subfamilies, see the concluding remarks of Section 3.

In the theory developed in the finite case, after dichotomies like above established, one refines the information by using Ramsey theory on trees in order to get canonical objects in a stronger sense [6]. One would like to have an embedding σ whose behavior is not only known on p-chains but also on the other different kinds of chains and antichains that naturally exist on the tree. However, the structure of chains and antichains in $\omega^{<\omega}$ and Υ is substantially more complex than in an n-adic tree $n^{<\omega}$ and we do not see how to apply the Ramsey refinement procedure to obtain a list of minimal canonical objects in the same elegant way as in the finite case.

A compact space is called Rosenthal if it is homeomorphic to a pointwise compact set of functions of the first Baire class on a Polish space. This is an important notion with origins in Banach space theory [14, 13], very much connected to descriptive set theory [2, 9]. The analytic gap dichotomies are related to finding canonical objects inside certain classes of separable Rosenthal compact spaces related to trees [16, 1, 7]. The following is the infinite version of [7, Lemma 35] in the spirit of this note. Here, $[\Upsilon] = \{x \in \omega^\omega : x_i \leq i \text{ for all } i < \omega\}$, that we can view as the set of branches of the tree Υ.

THEOREM 1.6. *Let K be a separable Rosenthal compactum. One and only one of the two following statements holds*

(i) *Either K is a continuous image of a compact finite-to-one preimage of a metric compactum.*

(ii) *Or there exists a one-to-one map $u : \Upsilon \to K$ such that for every $x \in [\Upsilon]$ and $p < \omega$ we have*
$$\overline{u(a_p^x)} \cap \overline{\bigcup_{q \neq p} u(a_q^x)} = \emptyset.$$
Moreover, we can suppose that $u(a_p^x)$ is a convergent sequence whenever a_p^x is infinite.

In Section 2 we will study the two notions of countable separation and their relation. Section 4 contains the proof of the main results, that will make use of games similarly as it is done in [4, 7] in the case of finitely many ideals. In Section 5 we deal with Rosenthal compact spaces.

2. General properties of countable separation

Proposition 2.1. *Consider* $\mathbb{I} \subset \mathcal{P}(\Omega)^\omega$ *and*

$$\mathbb{I}' = \{(a_0 \cup r_0, a_1 \cup r_1, \ldots) : (a_0, a_1, \ldots) \in \mathbb{I},\ r_0, r_1, \ldots \subset \Omega \text{ finite}\}.$$

The following are equivalent:

1. $\mathbb{I}$ *is* E*-countably separated,*
2. $\mathbb{I}'$ *is* E*-countably separated,*
3. $\mathbb{I}'$ *is* ϵ*-countably separated.*

Proof. [1. $\Rightarrow$ 2.] If $\mathcal{G}$ witnesses E-countable separation of $\mathbb{I}$, then one has that $\mathcal{G}' = \{b \cup r : b \in \mathcal{G}, r \text{ finite}\}$ witnesses E-countable separation of $\mathbb{I}'$.

[2. $\Rightarrow$ 3.] Is evident as we already observed in the introduction.

[3. $\Rightarrow$ 1.] Let $\mathcal{F} \subset \mathcal{P}(\Omega)^\omega$ be a family that ϵ-countably separates $\mathbb{I}'$. Suppose that $\mathbb{I}$ is not E-countably separated. In particular, the family $\mathcal{G}$ of all finite modifications of terms of all sequences in $\mathcal{F}$ does not E-countably separate $\mathbb{I}$. This means that there exists $a = (a_0, a_1, \ldots) \in \mathbb{I}$ such that no sequence of the form $(b_0, b_1, \ldots, b_n, \Omega, \Omega, \Omega, \ldots)$ satisfies $b_i \in \mathcal{G}$, $a_i \subseteq b_i$ for $i = 0, \ldots, n$ and $b_0 \cap \cdots \cap b_n$ finite. Enumerate as $\{b^0, b^2, \ldots\}$ all sequences in $\mathcal{F}$ that witness ϵ-countable separation for a. That is, such that $a_i \subseteq b_i^k$ for all i and $\cap_{i<\omega} b_i^k$ finite. We know that for every k there are infinitely many i such that $b_i^k \neq \Omega$. Thus, we can construct $i_0 < i_1 < i_2 < \cdots$ and $n_0, n_1, n_2, \ldots \in \Omega$ such that $n_k \notin b_{i_k}^k$. Now consider

$$\tilde{a} = (a_0, a_1, \ldots, a_{i_1-1}, a_{i_1} \cup \{n_1\}, a_{i_1+1}, \ldots, a_{i_k-1}, a_{i_k} \cup \{n_k\}, a_{i_k+1}, \ldots),$$

the result of substituting a_{i_k} by $a_{i_k} \cup \{n_k\}$ in the sequence a, while leaving the rest of terms of the sequence a untouched. Notice that $\tilde{a} \in \mathbb{I}'$. However no sequence in $\mathcal{F}$ can witness ϵ-countable separation on $\tilde{a}$ and this is absurd. The proof is that if b witnesses this, then it also witnesses it for the *smaller* sequence a, so b must be one of the b^k. But $\tilde{a}_{i_k} \not\subseteq b_{i_k}^k$ because $n_k \notin b_{i_k}^k$. ∎

Notice that, with minor proof adjustment, Proposition 2.1 also holds true if we consider

$$\mathbb{I}'' = \big\{(a_0 \cup r_0, a_1 \cup r_1, \ldots) : (a_0, a_1, \ldots) \in \mathbb{I}, |r_i| \leq 1\big\} \quad \text{or}$$

$$\mathbb{I}''' = \big\{(a_0 \triangle r_0, a_1 \triangle r_1, \ldots) : (a_0, a_1, \ldots) \in \mathbb{I}, r_i \text{ finite}\big\}$$

instead of $\mathbb{I}'$.

There is a topological characterization of E-countable separation. In the following statement, a' denotes the set of accumulation points of a subset a of a topological space.

Proposition 2.2. *A family $\mathbb{I} \subset \mathcal{P}(\Omega)^\omega$ is E-countably separated if and only if there exists a compact metric space K that contains Ω such that for all $(a_0, a_1, \ldots) \in \mathbb{I}$ we have $\bigcap_{n<\omega} a'_n = \emptyset$. Moreover, one may suppose that all points of Ω are isolated in K.*

Proof. [$\Leftarrow$] Let $\mathcal{B}$ be a countable basis of the topology of K which is closed under finite unions and intersections. We claim that

$$\mathcal{G} = \big\{(B \cap \Omega) \cup r \,:\, B \in \mathcal{B}, r \subset \Omega \text{ finite}\big\}$$

is the separating family that we need. Indeed, if $(a_0, a_1, \ldots) \in \mathbb{I}$, then by hypothesis $\bigcap_{n<\omega} a'_n = \emptyset$, and by compactness there exists n such that $a'_0 \cap \cdots \cap a'_n = \emptyset$. An easy inductive argument using normality [5, Lemma 9] shows that there exist $B_0, \ldots, B_n \in \mathcal{B}$ such that $a'_k \subset B_k$ for all k and $\overline{B_0} \cap \cdots \cap \overline{B_n} = \emptyset$. Since the B_k are open, there are finite sets $r_k \subset a_k$ such that $a_k \subset B_k \cup r_k$, so that $a_k \subset (B_k \cap \Omega) \cup r_k$. We found the separating sets $(B_k \cap \Omega) \cup r_k$, for $r = 0, \ldots, n$, that we were looking for.

[$\Rightarrow$] Let $\mathcal{G} \subset \mathcal{P}(\Omega)$ be the family that witnesses E-countable separation. By adding the finite sets and taking the subalgebra generated, we can suppose that $\mathcal{G}$ is a countable algebra of subsets of Ω that contains the finite sets. Let K be the Stone space of $\mathcal{G}$. Each element of Ω can be identified with the corresponding principal ultrafilter in K, which is an isolated point. All required properties are satisfied. ∎

3. Critical examples

Proposition 3.1. *The family*

$$\mathbb{I}_\epsilon = \big\{(a^x_p)_{p<\omega} \,:\, x \in \omega^\omega\big\} \subset \big(\omega^{<\omega}\big)^\omega$$

is not ϵ-countably separated.

Proof. Assume, on the contrary, that the corresponding countable family $\mathcal{F}$ exists. For every $k < \omega$ consider the set $T[k] = k^{<k}$. This is a tower of finite sets that cover $\omega^{<\omega}$. For every $x \in \omega^\omega$ there must exist $b^x = (b^x_p)_{p<\omega} \in \mathcal{F}$ and $k^x < \omega$ such that $\bigcap_p b^x_p \subset T[k^x]$ and $a^x_p \subseteq b^x_p$ for all p. By the Baire category theorem there must exist $b \in \mathcal{F}$ and $k < \omega$ such that the interior of the closure of $\{x : b^x = b, k^x = k\}$ is nonempty. This means that there exists $s \in \omega^{<\omega}$ that we can take of length $\geq k$ such that for every $t \in \omega^{<\omega}$ that end-extends s there exists $y \in \omega^\omega$ that end-extends t such that $b^y = b$ and $k^y = k$. For each $p < \omega$, if t is chosen as s followed by p then the corresponding y will satisfy $s \in a^y_p \subseteq b^y_p = b_p$. This contradicts that $\bigcap_p b_p \subset T[k]$. ∎

It follows form the above proposition that the two conditions of Theorem 1.4 are incompatible. So this theorem will mean that $\mathbb{I}_\epsilon$ is, in a sense, a minimal example of an analytic hereditary family of sequences that is not ϵ-countably separated. In an analogous way the following proposition implies that the two conditions of Theorem 1.5 are incompatible, and this theorem will mean that $\mathbb{I}_E$ is a minimal example of an analytic hereditary family of sequences that is not E-countably separated.

Proposition 3.2. *The family*

$$\mathbb{I}_E = \big\{(a^x_p)_{p<\omega} : x \in [\Upsilon]\big\} \subset \mathcal{P}(\Upsilon)^\omega$$

is ϵ-countably separated but not E-countably separated.

Proof. In order to show that $\mathbb{I}_E$ is not E-countably separated, by Proposition 2.1, it is enough to show that the family $\mathbb{I}'_E$ of finite modifications of $\mathbb{I}_E$ is not ϵ-countably separated. This is the same as to show that the hereditary family

$$\mathbb{I}''_E = \big\{(y_p)_{p<\omega} : \exists\, x \in \mathbb{I}'_E \text{ such that } y_k \subseteq x_k \text{ for all } k\big\}$$

is not ϵ-countably separated. By Proposition 3.1, it is enough to define an injective mapping $\sigma : \omega^{<\omega} \to \Upsilon$ such that $(z_p)_{p<\omega} \in \mathbb{I}_\epsilon$ implies $(\sigma(z_p))_{p<\omega} \in \mathbb{I}''_E$. For this, we want that, for every $x \in \omega^\omega$ there exists $\bar\sigma(x) \in [\Upsilon]$ such that $\sigma(a^x_p)$ is contained in a finite modification of $a^{\bar\sigma(x)}_p$ for every $p \in \omega$. Given $s = (s_0, s_1, s_2 \ldots, s_n)$ consider $t_i = \min\{i, s_i\}$ and then define $\sigma(s)$ as

$$(t_0, t_0{+}1, t_0{+}2, \ldots, s_0{-}1, s_0, s_0, t_1, t_1{+}1, t_1{+}2, \ldots, s_1, s_1, t_2, t_2{+}1, \ldots, s_n, s_n).$$

Similarly, if $s = (s_1, s_2, \ldots) \in \omega^\omega$, then

$$\bar\sigma(s) = (t_0, t_0{+}1, t_0{+}2, \ldots, s_0{-}1, s_0, s_0, t_1, t_1{+}1, t_1{+}2, \ldots, s_1, s_1, t_2, t_2{+}1, \ldots).$$

This works. Indeed, $\sigma(s) \in \Upsilon$ because the definition of the numbers t_i and the delays introduced in the definition of $\sigma(s)$ ensure that $\sigma(s)_i \leq i$. And if $x = (s_0, s_1, s_2, \ldots)$, then

$$\sigma(a_p^x) = \{(t_0, t_0+1, \ldots, s_0-1, s_0, s_0, t_1, t_1+1, \ldots, s_n, s_n) : s_{n+1} = p\} \subset a_p^{\bar{\sigma}(x)}.$$

It remains to check that $\mathbb{I}_E$ is ϵ-countably separated. We will prove the stronger fact that the family

$$\mathbb{I} = \left\{(a_p)_{p<\omega} \,:\, a_p \text{ is a } p\text{-chain for every } p\right\}$$

is ϵ-countably separated. Let b_p be the set of all elements of Υ of length at least p and let $\mathcal{F}$ be the family of sequences consistiting of $(b_0, b_1, b_2, b_3, \ldots)$ and all sequences of the form $(\Upsilon, \Upsilon, \Upsilon, \Upsilon, \ldots, \Upsilon, \{t\}, \Upsilon, \Upsilon, \ldots)$, all entries Υ except one that is a singleton. This family witnesses ϵ-countable separation. All these sequences have a finite intersection. If $(a_0, a_1, \ldots) \in \mathbb{I}$ there are two options. If the p-chain a_k is a singleton for some k, then it is dominated by $(\Upsilon, \Upsilon, \ldots, \Upsilon, a_k, \Upsilon, \Upsilon, \Upsilon, \ldots) \in \mathcal{F}$. This also works if one a_k is empty. On the other hand, if each of the p-chains $a_p \subset \Upsilon$ has at least two elements, then $a_p \subset b_p$ for all p. ∎

Suppose that we have families of sets $\{\mathcal{I}_i\}_{i<\omega}$ such that $\{\mathcal{I}_i\}_{i<n}$ is countably separated for some $n < \omega$. It follows easily from the definition that the families $\{\mathcal{I}_i\}_{i<\omega}$ are then ϵ-countably separated. We finish this section by observing that the converse is not true, and therefore the new notions of countably separation cannot be reduced to countable separation of finitely many families. The example is the following: For every $k < \omega$ take

$$\tilde{T}_k = \{(k, t_1, \ldots, t_n) \,:\, t_i < k, i = 1, \ldots, n\}$$

and $\tilde{T} = \bigcup_{k=1}^{\infty} \tilde{T}_k$. This would be just the disjoint union of incomparable copies of the trees $k^{<\omega}$. Let $\mathcal{I}_p$ be the set of p-chains inside $\tilde{T}$. $\{\mathcal{I}_p\}_{p<n}$ fails to be countably separated for every p, as $\tilde{T}$ contains a natural copy of $n^{<\omega}$. However the families $\{\mathcal{I}_p\}_{p<\omega}$ are E-countably separated. Indeed, it is enough to take $\mathcal{G} = \{\tilde{T}_k : k < \omega\} \cup \{\{t\} : t \in \tilde{T}\}$. If a_p is a p-chain for every p, then for every p there is k_p such that $a_p \subset \tilde{T}_{k_p}$. If some a_p is a singleton, we can take $b_p = a_p$. If none is a singleton then $k_p \geq p$ for all p, and in particular there exist $p_1 < p_2$ such that $k_{p_1} \neq k_{p_2}$. We take $b_p = \tilde{T}_{k_p}$ for $p = 0, 1, \ldots, p_2$.

4. The two dichotomies

In this section we write the proof of Theorem 1.4. The proof of Theorem 1.5 is just the same making some obvious modifications, forcing Player II in the game described below to play $p_i \leq i$ for all i. We have already shown that the two options of the dichotomy are incompatible, so our aim is to check that at least one of the options must hold.

Let $f : \omega^\omega \to \mathbb{I}$ be a continuous surjection. We consider a game where, at each turn $i < \omega$, Player I picks $n_i \in \Omega \setminus \{n_j : j < i\}$ and $\xi_i < \omega$ and then Player II picks $p_i < \omega$. Player I wins if and only if $\{n_i : p_i = p\} \subseteq f((\xi_i)_{i<\omega})_p$ for every p. This is a Borel game, so it is determined.

We first prove that if Player I has a winning strategy then the first option holds. A first attempt would be to define $\sigma(s)$ as the element of Ω that the strategy of Player I dictates to play after Player II has chosen the entries of s. The only problem is that such σ would not be injective, and for this we have to refine this idea a little bit. We define σ inductively with respect to an enumeration of $\omega^{<\omega}$ compatible with end-extension (s is enumerated after t whenever s is an end-extension of t), together with an auxiliary function $\tau : \omega^{<\omega} \to \omega^{<\omega}$. In the initial case, $\tau(\emptyset) = \emptyset$ and $\sigma(\emptyset)$ is the element of Ω that Player I plays as initial move. Take $s \in \omega^{<\omega}$, $s \neq \emptyset$ and we define $\sigma(s)$ and $\tau(s)$ assuming that they are already defined for previous finite sequences. Let p be the last coordinate of s, so that $s = (s_0, \ldots, s_m, p)$ and let $s^- = (s_0, \ldots, s_m)$ be the result of removing this last coordinate. For every $t < \omega$ let n_t and ξ_t be what the strategy of Player I dictates to play after Player II plays $\tau(s^-)$ followed by $p, 1, 2, 3, 4, \ldots, t$. Since Player I is not allowed to repeat, there must be a t such that n_t is different from all the values of σ previously defined. We define $\sigma(s)$ to be that n_t and $\tau(s)$ to be $\tau(s^-)$ followed by $p, 1, 2, 3, \ldots, t$. In this way σ is clearly injective. If we have a p-chain c_p inside a branch $B = (q_0, q_1, q_2, \ldots)$ of $\omega^{<\omega}$, notice that $\{\tau(s) : s \in B\}$ lies in another branch of $\omega^{<\omega}$ of the form

$$(q_0, 1, 2, 3, \ldots, t_0, q_1, 1, 2, 3, \ldots, t_1, q_2, 1, 2, 3, \ldots).$$

If Player II plays the above sequence of integers and Player I plays according to its strategy, a run of the game will be generated, where Player I will play $\sigma(q_0, \ldots, q_{k-1})$ (and some integer) before Player II plays q_k. If $\xi \in \omega^\omega$ is the sequence formed by the second choices of Player I along the run, we will have, since Player I wins, that

$$\{\sigma(q_0, \ldots, q_{k-1}) : q_k = p\} \subseteq f(\xi)_p.$$

Since c_p is a p-chain lying in $(q_0, q_1, \ldots)$, then $c_p \subseteq \{(q_0, \ldots, q_{k-1}) : q_k = p\}$. Therefore $\sigma(c_p) \subseteq f(\xi)_p$. Since $f(\xi) \in \mathbb{I}$ and $\mathbb{I}$ is hereditary we are done. Now suppose that Player II has a winning strategy and we shall see that the second option holds. Let $\mathcal{S}_{II}$ be the set of finite rounds of the game finishing at a movement of Player II that are played according to that winning strategy. Given $G = (n_1, \xi_1, p_1, \ldots, n_k, \xi_k, p_k) \in \mathcal{S}_{II}$ and given $\xi, p < \omega$, define

$$b^G_{\xi,p} = \{n \in \Omega \,:\, (n_1, \xi_1, p_1, \ldots, n_k, \xi_k, p_k, n, \xi, p) \not\in \mathcal{S}_{II}\}.$$

We claim that

$$\mathcal{F} = \left\{(b^G_{\xi,p} \cup \beta)_{p<\omega} \,:\, G \in \mathcal{S}_{II}, \xi < \omega, \beta \subset \Omega \text{ finite}\right\}$$

is the family we are looking for. First, notice that $\bigcap_{p<\omega} b^G_{\xi,p} = \emptyset$ for every G and ξ, so all intersections of sequences from $\mathcal{F}$ are finite. Suppose that we are given $(a_p)_p \in \mathbb{I}$. We can write $(a_p)_p = f((\xi_i)_{i<\omega})$. We suppose for contradiction that we could find no G and no ξ and no finite β such that $a_p \subset b^G_{p,\xi} \cup \beta$ for all p. Then we can play a full round of our game $(n_1, \xi_1, p_1, n_2, \xi_2, p_2, \ldots)$ according to the strategy of Player II in the following way. The ξ_i are always chosen as the coordinates of the ξ fixed before. If we have played $G_{i-1} = (n_1, \xi_1, p_1, \ldots, n_{i-1}, \xi_{i-1}, p_{i-1})$, then we can find p_i such that $a_{p_i} \not\subseteq b^{G_{i-1}}_{\xi_i,p_i} \cup \{n_1, \ldots, n_{i-1}\}$. Player I then plays $n_i \in a_{p_i} \setminus (b^{G_{i-1}}_{\xi_i,p_i} \cup \{n_1, \ldots, n_{i-1}\})$ and ξ_i, and then the strategy of Player II will be to play p_i (precisely because $n_i \not\in b^{G_{i-1}}_{\xi_i,p_i}$). This construction makes that $n_i \in a_{p_i}$ for all i, so $\{n_i : p_i = p\} \subseteq a_p = f((\xi_i)_{i<\omega})_p$ which contradicts that the strategy makes Player II win.

5. Rosenthal compact spaces

The following is the infinite version of [3, Theorem 7.2]

Lemma 5.1. *For a compact space K, the following are equivalent:*

(1) *K is a continuous image of a compact finite-to-one preimage of a metric compactum.*

(2) *There exists a countable family $\mathcal{F}$ of closed subsets of K such that for every infinite Y there exist finitely many sets from $\mathcal{F}$ whose union covers K but none of them covers Y.*

(3) *There exists a countable family $\mathcal{F}$ of closed subsets of K such that for every family $\{W_i\}_{i<\omega}$ of open sets with*

$$\overline{W_i} \cap \overline{\bigcup_{j\neq i} W_j} = \emptyset \qquad \textit{for all } i,$$

there exist finitely many sets in $\mathcal{F}$ that cover K, and such that none of the W_i's intersect all of these sets.

(4) *There exists a countable family $\mathcal{F}$ of closed subsets of K such that for every family $\{F_i\}_{i<\omega}$ of closed sets with*

$$F_i \cap \overline{\bigcup_{j\neq i} F_j} = \emptyset \qquad \textit{for all } i,$$

there exist finitely many sets in $\mathcal{F}$ that cover K, and such that none of the F_i's intersects all of these sets.

Proof. (1) $\Rightarrow$ (4): We suppose that there exist two continuous surjections $f : L \to K$ and $g : L \to M$ between compact spaces such that M is metric and $g^{-1}(x)$ is finite for all $x \in M$. Let $h : 2^\omega \to M$ be a continuous surjection from the Cantor set. Consider the basic clopen sets of the Cantor set $C_s = \{\sigma : \sigma|_n = s\}$, for $s \in 2^{<\omega}$ of length n. We claim that the family

$$\mathcal{F} = \left\{ f(g^{-1}(h(C))) \,:\, C \subseteq 2^\omega \text{ is clopen} \right\}$$

is as desired. Take $\{F_i\}_{i<\omega}$ as in (4).

Claim 1. Fore every $\sigma \in 2^\omega$ there exist $n < \omega$ and $i < \omega$ such that $f^{-1}(F_i) \cap g^{-1}(h(C_{\sigma|_n})) = \emptyset$. Proof of the claim: Otherwise, by compactness, there would exist σ such that for every $i < \omega$ we would have

$$\begin{aligned}\emptyset &\neq f^{-1}(F_i) \cap \bigcap_n g^{-1}(h(C_{\sigma|_n}))\\ &= f^{-1}(F_i) \cap g^{-1}\Big(\bigcap_n h(C_{\sigma|_n})\Big) = f^{-1}(F_i) \cap g^{-1}(h(\sigma)),\end{aligned}$$

which contradicts that $g^{-1}(h(\sigma))$ is finite.

As a consequence of Claim 1,

$$\left\{ C_s \,:\, \exists\, i < \omega \;\, f^{-1}(F_i) \cap g^{-1}(h(C_s)) = \emptyset \right\}$$

is an open covering of 2^ω, hence it has a finite subcover $\{C_s : s \in S\}$. Then

$$\bigcup_{s\in S} f\big(g^{-1}(h(C_s))\big) = f\Big(g^{-1}\Big(h\Big(\bigcup_{s\in S} C_s\Big)\Big)\Big) = K.$$

For every $i < \omega$, define

$$A_i = \bigcup \big\{ f\big(g^{-1}(h(C_s))\big) \ : \ s \in S,\ f(g^{-1}(h(C_s))) \cap F_i = \emptyset \big\}.$$

Since S is finite, the family $\{A_i : i < \omega\} \subset \mathcal{F}$ is finite even if parametrized by i. They cover K because, by the definition of the open covering, for every $s \in S$ there exists i such that $F_i \cap f(g^{-1}(h(C_s))) = \emptyset$. Each A_i is disjoint with F_i.

(4) $\Rightarrow$ (3): Is obvious.

(3) $\Rightarrow$ (2): Take $Y \subseteq K$ infinite. It is enough to find open subsets W_i of K as in (3) that all intersect Y. This is an elementary exercise in topology. We can suppose that Y is countable. By Urysohn's lemma, for every $y_0 \neq y_1$ in Y there is a continuous function $h_{y_0,y_1} : K \to [0,1]$ with $h(y_k) = k$ for $k = 0,1$. Putting all these functions together, we have a continuous function $h : K \to [0,1]^{\{(y_0,y_1)\in Y^2 : y_0 \neq y_1\}}$ that has metrizable range and is injective on Y. Since the range is a metrizable compactum, we can find an infinite sequence $\{y_n\}_{n<\omega}$ such that $\{h(y_n)\}_{n<\omega}$ converges to a point $z \notin \{h(y_n)\}$. Fixing a metric d and passing to a subsequence we can suppose that the sequence of distances $r_n = d(h(y_n), z)$ is strictly decreasing, and then define the desired open sets as $W_n = \{y \in K : d(h(y), z) \in (r_{3n-1}, r_{3n+1})\}$.

(2) $\Rightarrow$ (1): Now suppose that $\mathcal{F}$ is a family of closed sets like in (2). Consider the countable family Z of all the finite subsets F of $\mathcal{F}$ whose union is K. The following set

$$L = \left\{ (x, (A_F)_{F\in Z}) \in K \times \prod_{F\in Z} F \ : \ x \in \bigcap_{F\in Z} A_F \right\}$$

is a closed, therefore compact set. Here $\prod_{F\in Z} F$ is endowed with the product topology of the discrete topology in each finite set F. The projection on the first coordinate $L \to K$ is onto. Condition (2) implies that the projection on the second coordinate $L \to \prod_{F\in Z} F$ is finite-to-one. ∎

We proceed now to the proof of Theorem 1.6. We show first that the two conditions are incompatible. Suppose that both hold. Take a family $\mathcal{F}$ like in Lemma 5.1 (4). We claim that the family $\mathcal{G} = \big\{u^{-1}(K \setminus F) : F \in \mathcal{F}\big\}$

witnesses the E-countable separation of $\mathbb{I}_E$ in Proposition 3.2, a contradiction. For every $x \in [\Upsilon]$, we apply Lemma 5.1 (4) to the closed sets $\overline{u(a_p^x)}$ for $p < \omega$, obtaining finitely many sets $F^1, \ldots, F^m \in \mathcal{F}$ that cover K and so that no $\overline{u(a_p^x)}$ intersects all F^j. This implies that

$$\bigcap_{j=1}^{m} u^{-1}(K \setminus F^j) = \emptyset,$$

and for every p there exists $j \in \{1, \ldots, m\}$ such that $a_p^x \subset u^{-1}(K \setminus F^j)$, which is what is required for E-countable separation. Now we prove that one of the options must hold. We suppose that K is a Rosenthal compact and D is a countable dense subset of K. Consider the space $C_D(K)$ of bounded real valued continuous functions of K endowed with the topology of pointwise convergence of K. This is an analytic space by a result of Godefroy [10], so we fix a continuous surjection $\pi : \omega^\omega \to C_D(K)$.

We play a game of length ω.

At stage k Player I chooses $d_k \in D \setminus \{d_0, \ldots, d_{k-1}\}$ together with natural numbers s_{ij} for $i, j < \omega$, $\max(i, j) = k$. Player II responds with a natural number $p_k \leq k$. At the end of the game, consider $s_i = (s_{i0}, s_{i1}, s_{i_2}, \ldots) \in \omega^\omega$ for every $i < \omega$. Player I wins if for every $i < \omega$ we have

$$\pi(s_i)(d_n) < 0 < 1 < \pi(s_i)(d_m) \qquad \text{whenever } p_n = i \text{ and } p_m \neq i.$$

This is a Borel game. By Martin's theorem, one of the two players must have a winning strategy.

First, we prove that a winning strategy for Player I will give condition (ii). As a first try, we can define $u(p_0, \ldots, p_n)$ as the element d_{n+1} given by the strategy of Player I if Player II plays $p_0, \ldots, p_n$. This u would satisfy the condition that

$$\overline{u(a_p^x)} \cap \overline{\bigcup_{q \neq p} u(a_q^x)} = \emptyset$$

for all $x \in [\Upsilon]$ and $p < \omega$, but it may fail to be one-to-one. However, it is easy to inductively refine this to make u injective, in a similar way as we did at the beginning of Section 4: If $u'(p_0, \ldots, p_n) = u(t)$ is already defined, then for each p define $u'(p_0, \ldots, p_n, p) = u(t^\frown p, 0, 1, 2, 3, \ldots, n)$ for n high enough to make sure that the value is different from all the $u'(s)$ whose value has been already established. It remains to justify that we can assume that $u(a_p^x)$ is a convergent sequence whenever a_p^x is infinite. So we assume that have u as in condition (ii) of the theorem, and we will find a new u' with that extra condition. We will define $u'(s)$ together with a strong subtree T_s of Γ in such a way that $T_t \subset T_s$ if $s < t$ and $u'(s)$ is the image under u of the root of T_s. A strong subtree of Υ is a subset $T \subset \Upsilon$ with a root (minimum for $<$) and the property that every immediate successor in Υ of a node $t \in T$ has exactly one extension in the next level of Υ that has nonempty intersection with T, and conversely every node of T that is not the root is the extension of a node in the previous level of Υ that intersects T. A level is the set of nodes that have a given length. We are using Milliken's theory [12], cf. [17, Chapter 3]. As a starting point, we declare $u'(\emptyset) = u(\emptyset)$ and $T_\emptyset = \Upsilon$. Now, suppose that $u'(s)$ and T_s have been already defined for all s of length n, and let $s^\frown i$ be of length $n+1$. We color the infinite n-chains of $\{t \in T_s : t \geq u'(s)^\frown i\}$ with two colors, depending on whether their image under u is convergent or not. Since K is a Rosenthal compactum, the family of convergent subsequences of $u(T_s)$ is a conalytic family of subsets of $u(T_s)$, cf. the beginning of the proof of [7, Lemma 36]. This means that that coloring is Souslin measurable and we can apply Milliken's Theorem [17, Theorem 6.13], so there is a strong subtree $T_{s^\frown i} \subset \{t \in T_s : t \geq u'(s)^\frown i\}$ where all n-chains have the same color. This color must be that all sequences are convergent, because Rosenthal compact spaces are sequentially compact [14] (even Fréchet-Urysohn, by the celebrated result of Bourgain, Fremlin and Talagrand [8]). We will define $u'(s^\frown i)$ as the image under u of the root of $T_{s^\frown i}$.

Now we suppose that Player II has a winning strategy and we will prove condition (i). Given a finite round of the game ξ where Player I and II have played k movements each, given $s = (s_{ij})_{\max(i,j)=k}$, and $p \leq k$ we define $F_p(\xi, s)$ to be the set of all $d \in D$ such that the strategy of Player II gives p after ξ is played and then Player I plays (d, s). Let $\mathcal{F}$ be the family of all sets of the form $\overline{F_p(\xi, s)}$ together with all singletons from D. Let $\{W_i : i < \omega\}$ be an infinite family of open sets as in Lemma 5.1 (3) and let us check that the statement (3) in that lemma holds. For every i, let ψ_i be a real-valued continuous function on K that is negative on W_i and greater than one on all W_j with $j \neq i$. Each ψ_i is of the form $\psi_i = \pi(s_{i0}, s_{i1}, \ldots)$.

Suppose that Player I always plays those s_{ij} and elements $d_k \in W_{p_k}$. If Player II always responds with that p_k, then he will lose. That means that, in particular, there must be a finite round of the game ξ of length k such that whenever Player I tries to play next some $d_k \in W_q$ and the s_{ij}, then Player II will always respond with some $p \neq q$. In other words, that means that $W_q \cap F_q(\xi, \{s_{ij}\}) = \emptyset$ for every q. Since W_q is open, this implies that $W_q \cap \overline{F_q(\xi, \{s_{ij}\})} = \emptyset$. We also have that

$$\bigcup_q \overline{F_q(\xi, \{s_{ij}\})} = \overline{\bigcup_q F_q(\xi, \{s_{ij}\})} = \overline{D \setminus D_0} \supseteq K \setminus D_0.$$

Here, D_0 is the finite set of all $d_i \in D$ that have been played by Player I along ξ. The parameter q in the union above takes only finitely many values, at most the length of the partial game ξ. So we have found finitely many closed sets from our family $\mathcal{F}$ as desired.

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