EXTRACTA MATHEMATICAE

Article in press
Available online May 7, 2025

Isocanted cube: the lower Lebesgue volumes, incidence numbers and symmetries of this d–dimensional tile

M.J. de la Puente

Departamento de Álgebra, Geometría y Topología, Facultad de Matemáticas
Universidad Complutense, 28040 *Madrid, Spain*

mpuente@ucm.es

Received March 13, 2025
Accepted April 8, 2025

Presented by V. Soltan

Abstract: Let $d \geq 2$. In this paper we prove that $\mathcal{I}_d(\ell, a)$ fills $\mathbb{R}^d$ face–to–face by translations. We prove that the symmetry group of $\mathcal{I}_d(\ell, a)$ contains the product of cyclic groups $C_d \times C_2$ as a subgroup. We compute the Lebesgue j–volume (i.e., the sum of the Lebesgue j–measures of the j–faces) of $\mathcal{I}_d(\ell, a)$, for $1 \leq j < d$. We compute the incidence numbers (as defined by Grünbaum) of the faces of $\mathcal{I}_d(\ell, a)$.

Key words: tile space, symmetry group, face volume, cubical zonotope, isocanted cube, incidence number, perturbation.

MSC (2020): 52C22, 52B15, 52A38.

1. Introduction

Cubes, simplices, zonotopes and cyclic polytopes, as well as their polar duals, are common examples of convex polytopes. The metric properties of regular (i.e. Platonic) and Archimedean polytopes, as well as pyramids have been settled. Some combinatorial properties of 0–1 polytopes, neighborly polytopes, simplicial and cubical polytopes have been widely studied. Yet, survey papers still ask for more examples of families of convex polytopes (cf. [45]). In this paper we continue the study of the family $\mathcal{I}_d(\ell, a)$, the *isocanted cubes* (cf. [33, 34]). $\mathcal{I}_d(\ell, a)$ is the convex polytope defined by the following inequalities in $\mathbb{R}^d$:

$$-\frac{\ell}{2} \leq x_j \leq \frac{\ell}{2}, \quad a-\ell \leq x_j - x_k \leq \ell - a, \qquad \forall\, j,k \in [d],\ j \neq k. \tag{1.1}$$

Let $\mathcal{C}_d(\ell)$ denote the d–cube of edge–length $\ell > 0$, centered at the origin $\mathbf{o} \in \mathbb{R}^d$ and having edges parallel to the coordinate axes. Everyone knows

ISSN: 0213-8743 (print), 2605-5686 (online)

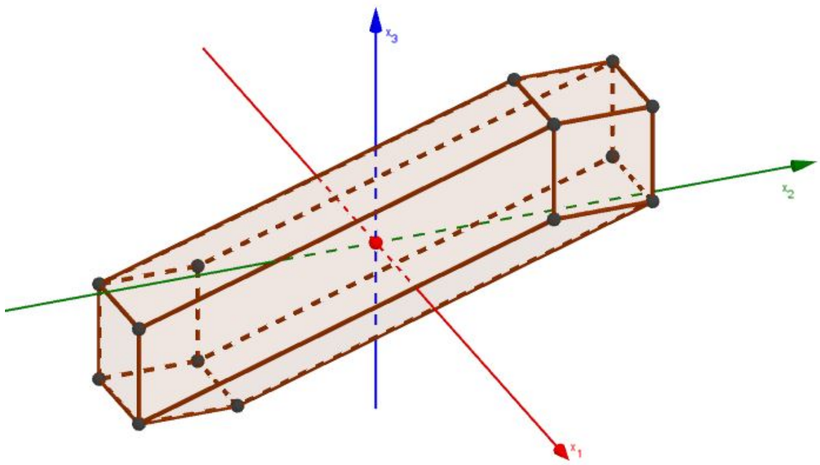

Figure 1: Isocanted cube ($d = 3$) with $\ell < 2a$

that the d–cube fills $\mathbb{R}^d$ face–to–face by translations. $\mathcal{I}_d(\ell, a)$ is a certain uniform perturbation of $\mathcal{C}_d(\ell)$ where a is a real number with $0 < a < \ell$. Here we prove that $\mathcal{I}_d(\ell, a)$ also fills $\mathbb{R}^d$ face–to–face by translations. Thus $\mathcal{I}_d(\ell, a)$ is a *tile* (some authors say it is a *parallelotope*.[1])

This result has a number of consequences. First, $\mathcal{I}_d(\ell, a)$ is *affinely equivalent* to the *Voronoi cell* of a certain lattice in $\mathbb{R}^d$, (i.e., finitely generated additive subgroup of $\mathbb{R}^d$) by *Erdahl's Theorem* (cf. [17]). Second, $\mathcal{I}_d(\ell, a)$ is *translations scissors congruent* to a d–cube (i.e., $\mathcal{I}_d(\ell, a)$ can be decomposed into a finite number of polytopes which, only using translations, can be reassembled to produce a cube), by *Mürner's Theorem* (cf. [32]). $\mathcal{I}_d(\ell, a)$ can be viewed as a certain uniform *perturbation* of $\mathcal{C}_d(\ell)$ and the limit of $\mathcal{I}_d(\ell, a)$, as a tends to zero, is $\mathcal{C}_d(\ell)$ (cf. Figure 1). In [34] we computed the volume of $\mathcal{I}_d(\ell, a)$. In this paper we compute the j–th volume of $\mathcal{I}_d(\ell, a)$ (i.e. the sum of the Lebesgue volumes of its j–faces), for $1 \leq j < d$. In [34] we proved (incompletely) that lattice of proper faces of $\mathcal{I}_d(\ell, a)$ (i.e., partially ordered set having unique least upper bounds and greatest lower bounds) is isomorphic to the lattice of proper subsets of $[d] \cup \{0\}$. In this paper we complete that proof, and compute more combinatorial invariants of $\mathcal{I}_d(\ell, a)$, namely, the *incidence numbers* of its faces (according to Grümbaum's definition, the j–th

[1]The term parallelotope can also mean the generalization of parallelograms to higher dimensions, i.e., the image of a cube under a bijective affine–linear map.

incidence number of a face F in P is the number of faces of P of dimension j incident to F). In our proofs, we use the following known facts: (1) $\mathcal{I}_d(\ell, a)$ is a cubical zonotope having $d+1$ zones: this is easy to see, since $\mathcal{I}_d(\ell, a)$ is the result of moving $\mathcal{C}_d(\ell)$ along a segment in the direction of the all–ones vector (cf. Figure 1). (2) The definition of $\mathcal{I}_d(\ell, a)$ is via $2d + 2\binom{d}{2} = (d+1)d$ supporting hyperplanes. Remarkably, passing to the vertex–enumeration of $\mathcal{I}_d(\ell, a)$ is possible, because in $\mathcal{I}_d(\ell, a)$ some vertices are more relevant than the rest. Indeed, each vertex in $\mathcal{I}_d(\ell, a)$ can be expressed in terms of a unique subfamily of $d+1$ vertices, called the *generators* of $\mathcal{I}_d(\ell, a)$ (more precisely, each point in $\mathcal{I}_d(\ell, a)$ is tropically spanned by its generators, cf. [33]). (3) There exists a rotation in $\mathbb{R}^d$ around a 1–dimensional axis that takes generators to generators (except for one generator, which remains fixed). (4) After assigning to the generators labels in the set $[d] \cup \{0\}$, the remaining vertices and the faces of $\mathcal{I}_d(\ell, a)$ can be labeled in a combinatorial fashion (showing incidence). Our labelling is denoted $\mathcal{L}$. (5) The set of facets of $\mathcal{I}_d(\ell, a)$ splits into three subsets: North Cask, South Cask and Equator.

Why do we care about $\mathcal{I}_d(\ell, a)$? The polytope $\mathcal{I}_d(\ell, a)$ is connected to Linear Algebra and Functional Analysis. Indeed, let I_d and J_d denote the identity matrix and the all–ones matrix of size d, respectively, and define the *Bose–Mesner matrix* as $M_d(\ell, a) := \ell I_d + (\ell - a)J_d$. $M_d(\ell, a)$ is a positive, rank–one, uniform perturbation of ℓI_d. [2] Everybody knows that the d–volume of the cube $\mathcal{C}_d(\ell)$ is $\ell^d = \det \ell I_d$. Remarkably, the d–volume of $\mathcal{I}_d(\ell, a)$ is $\det M_d(\ell, a)$ (cf. expression (4.1), which was proved in [34]). Thus, one can say that determinants preserve the perturbation mentioned above in p. 2.[3] In Functional Analysis, some Lipschitz–free Banach spaces (associated to a finite metric space having $d+1$ elements) have unit ball whose polar dual is equal to $\mathcal{I}_d(\ell, a)$ with $2a = \ell$ (cf. [2]).

$\mathcal{I}_d(\ell, a)$ has connections to Mechanical Engineering and Machine Learning. Indeed, some sections of the *Tresca–Guest yield surface* are equal to $\mathcal{I}_2(\ell, a)$. Also, in the attempt to steal information from *large language models* (LLM), convex polytopes slightly more general than isocanted cubes have been used (cf. algorithm 3 in [12]).

Where does $\mathcal{I}_d(\ell, a)$ appear in the family picture of zonotopes?[4] Isocanted

[2]The chosen name honors the mathematician R.C. Bose, who introduced this matrix in 1949, in a paper about *balanced block designs* (cf. [10, 11]). Bose is widely known by the *Bose-Chaudhuri–Hocquenghem codes* (BCH codes), and the *Bose-Mesner algebras* (certain unitary commutative algebras of matrices).

[3]The Bose–Mesner matrix is rather ubiquitous! (cf. in [12, p. 20] and in [47, p. 190]).

[4]Zonotopes are sums of segments, each segment giving rise to a zone. They have several

cubes belong to the class $\mathcal{Z}_{d,d+1}$ of d–dimensional zonotopes having $d+1$ zones. Even though $\mathcal{Z}_{d,d+1}$ has been studied, some metric problems concerning $\mathcal{Z}_{d,d+1}$ remain open (cf. [27]). Restricting to the cubical subclass $\mathcal{Z}^c_{d,d+1}$ (i.e., those having cubical facets), it is known that there is *just one combinatorial type* in $\mathcal{Z}^c_{d,d+1}$ (cf. [28]). Clearly, $\mathcal{I}_d(\ell,a)$ is a *realization* of this class. The proper faces of a d–zonotope can be labelled using as labels a subset $\mathcal{L}'$ of $\{+,-,0\}^p$, for some $p \geq d$ (cf. [44]). We think that our labelling $\mathcal{L}$ (mentioned above) is more useful than $\mathcal{L}'$. In a 1975 paper, McMullen characterizes those zonotopes T tiling $\mathbb{R}^d$ face–to–face by translations, and he describes the translation lattice $\Lambda(T)$ as the lattice spanned by the barycenters of the facets of T. In this paper we compute the barycenters for $T = \mathcal{I}_d(\ell,a)$ (they arise from the matrix $M_d(\ell,a)$). Then we prove that the determinant of a *fundamental parallelepiped* of $\Lambda(\mathcal{I}_d(\ell,a))$ equals the volume of $\mathcal{I}_d(\ell,a)$, getting a proof that $\mathcal{I}_d(\ell,a)$ is a tile.

What are the symmetries of $\mathcal{I}_d(\ell,a)$? This polytope is invariant by an axial rotation of angle $\frac{2\pi}{d}$, and by the antipodal map.

Why do we care about the incidence numbers of $\mathcal{I}_d(\ell,a)$? The *valency* (also called *degree*) of a vertex F in a polytope P is the number of edges of P meeting F. The f–vector of P is also combinatorial data about P. Both concepts were extended by Grünbaum to the so called *incidence numbers* of F in P (cf. [23]). We compute the incidence numbers of the faces F of $\mathcal{I}_d(\ell,a)$ and show that they depend not only on d and the dimensions of F, but also on the *minimal length* of F (i.e., the length of the vertex of F having minimal length in the labeling $\mathcal{L}$).

Why do we care about the lower Lebesgue volumes of $\mathcal{I}_d(\ell,a)$? Are the Lebesgue volumes the same as the intrinsic volumes? For a d–polytope P, the 1–volume is the sum of the lengths of the edges of P, the 2–volume is the sum of the areas of the 2–faces of P, etc. The *lower Lebesgue volumes* of P are

$$\operatorname{vol}_j(P) = \sum_{F \text{ face of } P \text{ of dim. } j} \operatorname{vol}_j(F), \qquad 1 \leq j \leq d.$$

where $\operatorname{vol}_j(F)$ denotes the Lebesgue measure in $\mathbb{R}^j$. The j–th Lebesgue volume

important characterizations. Zonotopes are (1) those polytopes whose 2–faces are zonotopes, and also (2) those polytopes whose 2–faces are centrally symmetric, as well as (3) those polytopes which are affine–linear projections of cubes. They are also characterized through the *Halwka inequality* and the *7–polygonal inequality* (cf. [22]). For more on zonotopes, cf. [4, 13, 25, 27, 28, 39], and on tiling zonotopes cf. [16, 29, 30, 31, 42] and the many references therein.

of P is different from the j–th intrinsic volume of P, denoted $V_j(P)$. Indeed,

$$V_j(P) = \sum_{F \text{ face of } P \text{ of dim. } j} \gamma(F,P)\,\mathrm{vol}_j(F), \qquad 1 \le j \le d$$

where $\gamma(F,P)$ is the *(normalized) external angle* of P at F (see [24, 43]).

The paper is organized as follows. Section 2 contains background. In subsection 2.5 we recall the labelling $\mathcal{L}$ of vertices and faces of $\mathcal{I}_d(\ell,a)$. In subsection 3.3, we compute the Lebesgue j–volume of a isocanted cube of dimension d. It is a homogeneous polynomial of degree j in the variables ℓ, a, with coefficients in $\mathbb{Q}[\sqrt{d-j+1}]$. The proof takes into account that $\mathcal{I}_d(\ell,a)$ has two sorts of faces: Polar or Equatorial. In section 4 we prove, in two ways, that $\mathcal{I}_d(\ell,a)$ is a tile for $\mathbb{R}^d$. Our first proof consists on checking one of the six characterizations given by McMullen. Our second proof shows that $\mathcal{I}_d(\ell,a)$ provides a lattice packing with density equal to one. In section 5, we prove that $C_d \times C_2$ is a subgroup of the symmetry group of $\mathcal{I}_d(\ell,a)$. It is generated by a rotation about the North–South axis and the antipodal map $-\mathrm{id}$. In section 6 we compute the *incidence numbers* of the faces of $\mathcal{I}_d(\ell,a)$.

Notice that general isocanted cubes are *not rational polytopes*, unless the parameters a, ℓ, are rational. Do the vertices of $\mathcal{I}_d(\ell,a)$ form a *root system?* The answer is no, in general and yes, if $2a = \ell$ (the special case arising in Functional Analysis).

A word of advise. In [33] we studied isocanted alcoved polytopes. In [34] we continued the study of these polytopes, but we changed their name to *isocanted cubes*.

Notations and conventions: Tiling of $\mathbb{R}^d$ and filling of $\mathbb{R}^d$ are used indistinctly. $[d]$ denotes the set $\{1,2,\ldots,d\}$. $\binom{\alpha}{\beta} = 0$ whenever $\alpha, \beta \in \mathbb{N}$ and $\alpha < \beta$. Concerning f–vectors, $f_j(P) = 0$ for all $j < 0$ and all d–polytopes P (including $j = -1$, even if other authors set $f^d_{-1}(P) = 1$, accounting for the empty face of P). $v(P)$ is the set of vertices of P and the 0–volume of P is the cardinality of $v(P)$.

We let $u_d := \sum_{j=1}^d e_j$ denote the *all–ones vector*, where $(e_1, e_2, \ldots, e_d)$ is the *standard vector basis* in $\mathbb{R}^d$. We make the usual identification of d–tuples in $\mathbb{R}^d$ with vectors and/or points.

2. Background

For general facts on convexity, cf. [18, 19, 36, 40, 41, 43, 46]. See [5, 21, 23, 24, 44] for the definitions below. For more on polytopes, see [3, 6, 9, 14, 38].

2.1. On cubes and combinatorial cubes. $\Delta_1(\ell)$ is a (line) *segment* of length $\ell \geq 0$. Given $d \geq 0$, a *d–cube* of edge–length ℓ is the Cartesian product of d segments of equal length

$$\mathcal{C}_d(\ell) := \Delta_1(\ell) \times \overset{d)}{\cdots} \times \Delta_1(\ell).$$

A *combinatorial cube* (some authors call it a *cuboid* or a *cube*) is any polytope combinatorially equivalent to a cube. We denote it by $\mathcal{K}_d$. It is well–known that

$$f_j\left(\mathcal{K}_d\right) = 2^{d-j}\binom{d}{j}, \qquad j = 0, 1, \ldots, d. \tag{2.1}$$

2.2. On cubical polytopes and cubical complexes. A *polytopal complex* is a finite family of polytopes such that each face of a member of the family belongs to the family. A polytopal complex has dimension d if some of its maximal members have dimension d. For example, the *boundary complex* ∂P of a d–polytope P is a polytopal complex of dimension $(d-1)$.

Cubical polytopal complexes are, by definition, those whose maximal faces are all combinatorial cubes. *Cubical polytopes* are those whose facets are combinatorial cubes.

The many properties known for the family of *simplicial polytopes* yields to the hope that the family of *cubical polytopes* must have interesting properties on its own. The investigation of cubical polytopes began with Grümbaum, Perles and Shephard in the 1960's. Examples of certain cubical polytopes as well as some results are found in [23, Chapter 4]. In the 1980's, G. Blind and R. Blind continued the investigation, classification and construction of more cubical polytopes (cf. [7, 8]). In analogy with the definition of the h–vector and g–vector for simplicial polytopes, a search for a tailored definition of h–vector and g–vector for cubical polytopes began in [26] and [1]. A *cubical lower bound conjecture* has been stated and disproved.

2.3. On vertex casks of combinatorial cubes. Given a vertex F of a combinatorial d–cube $\mathcal{K}_d$, the *vertex cask of $\mathcal{K}_d$ at F*, denoted $\mathcal{V}_{d-1}$, is defined as the closure of the union of all the j–faces of $\mathcal{K}_d$ meeting F, for $j = 0, 1, 2, \ldots, d-1$. A vertex cask is a cubical complex of pure dimension $d-1$ and it is homeomorphic to a closed half–sphere. Its f–vector is (cf. [33, Theorem 3.7])

$$f_j\left(\mathcal{V}_{d-1}\right) = \left(2^{d-j} - 1\right)\binom{d}{j}, \qquad j = 0, 1, \ldots, d-2. \tag{2.2}$$

2.4. On zonotopes. The following is taken mainly from [13, 28, 39]. A *zonotope* is, by definition, a polytope $Z \subset \mathbb{R}^d$ obtained by *Minkowski sum* (i.e., vector sum) of finitely many (line) segments. The sum of the mid–points of the segments is the center of symmetry of Z. For metric issues, we can *assume* without loss of generality, the center of symmetry of a zonotope to be at the origin $\mathbf{o} \in \mathbb{R}^d$.

We consider r vectors $z_1, \ldots, z_r \in \mathbb{R}^d \setminus \{\mathbf{o}\}$ and corresponding segments $S_j = \operatorname{conv}(-z_j, z_j)$ and take the sum

$$Z = S_1 + \cdots + S_r. \tag{2.3}$$

Each segment S_j is called a *component* of Z. Each component of Z gives rise to a *zone* of Z: the j–th zone is, by definition, the complex of all faces of Z containing edges parallel to S_j. The name zonotope arises from this point of view.

We will *assume* that the vectors $z_1, \ldots, z_r$ span $\mathbb{R}^d$ and, moreover, that they are *generic* in the sense that the cardinality of the set

$$P(Z) := \{\pm z_1 \pm z_2 \pm \cdots \pm z_r\} \tag{2.4}$$

is at least $2^r - 1$. A point in $P(Z)$ can be either a *vertex* of Z, an *interior point* of Z (topologically) or neither, in which case we call it a *boundary point* of Z. The sets of such points are denoted $v(Z)$, $i(Z)$ and $b(Z)$ and so

$$P(Z) = v(Z) \,\dot\cup\, i(Z) \,\dot\cup\, b(Z).$$

Zonotopes can be cubical and, further, can be combinatorial cubes. A zonotope is a combinatorial cube (resp. cubical) if and only if it has neither interior nor boundary points (resp. no boundary points), (cf. [39, p. 310]). In symbols

$$P(Z) = v(Z) \qquad \text{if zonotope } Z \text{ is a } \mathcal{K}_d, \tag{2.5}$$

$$P(Z) = v(Z) \,\dot\cup\, i(Z) \qquad \text{if zonotope } Z \text{ is cubical}. \tag{2.6}$$

For the zonotope (2.3), any sum Z' of the form

$$\begin{aligned} Z' := S_{\sigma(1)} + \cdots + S_{\sigma(r')} &+ \epsilon_{\sigma(r'+1)} z_{\sigma(r'+1)} \\ &+ \epsilon_{\sigma(r'+2)} z_{\sigma(r'+2)} + \cdots + \epsilon_{\sigma(r)} z_{\sigma(r)} \end{aligned}$$

is called a *cell* of Z, where σ is a permutation of the set $[r]$, $\epsilon_j = \pm 1$ and $0 \le r' \le r$. Each cell of Z is a zonotope, clearly. Every face of Z is a cell of Z, but not conversely.

Lemma 2.1. (Face of cubical zonotope) *If Z is a cubical zonotope, then a cell Z' of Z is a face of Z if and only if $v(Z') \subseteq v(Z)$.*

Proof. For a cell Z' of Z we have $v(Z') \subseteq P(Z)$. Clearly, a face of Z is a cell of Z not meeting the interior of Z or, equivalently, not meeting the finite set $i(Z)$, i.e., $v(Z') \cap i(Z) = \emptyset$, which is equivalent to $v(Z') \subseteq v(Z)$, by (2.5). ∎

2.5. On isocanted cubes. Recall the inequalities (1.1). Clearly, $\mathcal{I}_d(\ell, a)$ is symmetric with respect to the origin $\mathbf{o}$ in $\mathbb{R}^d$. There exist *two special vertices* in $\mathcal{I}_d(\ell, a)$: $\mathbf{N}$ (resp. $\mathbf{S}$) is the coordinate–wise maximum (resp. minimum), called *North Pole* (resp. *South Pole*). In coordinates, we have $\mathbf{N} = \frac{\ell}{2} u_d$, $\mathbf{S} = -\mathbf{N}$, where u_d is the all–ones vector. This yields a description (and labeling) of faces of $\mathcal{I}_d(\ell, a)$. Alternatively, we have a description of $\mathcal{I}_d(\ell, a)$ as a zonotope. We us both descriptions in the coming paragraphs.

The properties below have been proved in [33, 34]:

$$\mathcal{I}_d(\ell, a) = S_1 + S_2 + \cdots + S_d + S_0$$

where

$$y_j = \frac{\ell - a}{2} e_j, \quad y_0 = \frac{a}{2} u_d \in \mathbb{R}^d, \quad S_j = \operatorname{conv}(-y_j, y_j), \quad j \in [d]. \tag{2.7}$$

Notice that the set $P(\mathcal{I}_d(\ell, a))$ introduced in (2.4) contains exactly 2^{d+1} elements if and only if $\ell \neq 2a$ and $2^{d+1} - 1$ otherwise.

$$f_j\left(\mathcal{I}_d(\ell, a)\right) = (2^{d+1-j} - 2)\binom{d+1}{j}, \quad j = 0, 1, \ldots, d-1. \tag{2.8}$$

This f–vector does *not* specialize to the f–vector of the cube, when $a = 0$.

Next we expand, rephrase and/or give more details to some statements from [33, 34].

The vertices of $\mathcal{I}_d(\ell, a)$ are obtained maximizing some variables and minimizing the rest, within the bounds given by (1.1). They receive a label according to this fact. Labels are proper subsets of $[d] \cup \{0\}$. For instance, if we maximize x_j and minimize the rest, we get a vertex of $\mathcal{I}_d(\ell, a)$ denoted $\underline{j}$. In symbols,

$$\underline{j} := \frac{\ell}{2} e_j + \frac{2a - \ell}{2} \sum_{j \neq k \in [d]} e_k.$$

More generally, given any $\emptyset \neq W \subseteq [d]$, if we maximize x_j with $j \in W$ and minimize the rest, we get a vertex of $\mathcal{I}_d(\ell, a)$ denoted $\underline{W}$:

$$\underline{W} := \frac{\ell}{2} \sum_{j \in W} e_j + \frac{2a - \ell}{2} \sum_{j \in [d] \setminus W} e_j. \tag{2.9}$$

We say that W is the label of the vertex $\underline{W}$. By central symmetry, the point $-\underline{W}$ is also a vertex of $\mathcal{I}_d(\ell, a)$. We check that these are all the vertices of $\mathcal{I}_d(\ell, a)$ by count: we have $2(2^d - 1) = 2^{d+1} - 2 = (2^{d+1-0} - 2)\binom{d+1}{0} = f_0(\mathcal{I}_d(\ell, a))$; see (2.8).

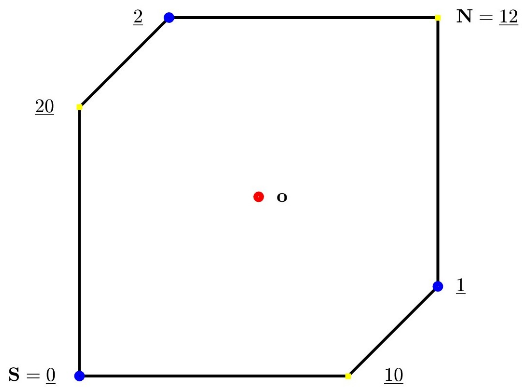

Figure 2: The labeling $\mathcal{L}$ of vertices in the isocanted square ($d = 2$) where the generators $\underline{0}$, $\underline{1}$, $\underline{2}$ are marked in blue. The origin is marked in red.

The following is a *injection* from the family of *non–empty* subsets of $[d]$ to $v(\mathcal{I}_d(\ell, a))$:

$$W \longmapsto \underline{W}$$

and it extends to a *bijection* $\mathcal{L}$ from $v(\mathcal{I}_d(\ell, a))$ to the family of *proper subsets* of $[d] \cup \{0\}$ by letting

$$\mathcal{L} : \quad -\underline{W} \longmapsto [d] \cup \{0\} \setminus W. \tag{2.10}$$

The bijection $\mathcal{L}$ is the *labeling of vertices* of $\mathcal{I}_d(\ell, a)$ given in [33] (cf. Figure 2). In particular, the South Pole $\mathbf{S}$ gets the label 0 because[5] $\mathbf{N}$ gets the label $[d]$ and $\mathbf{S} = -\mathbf{N}$.

[5]We have written 0 instead of $\{0\}$ for simplicity of notation. Sometimes we omit braces and commas when writing sets.

DEFINITION 2.2. The *length of a vertex* of $\mathcal{I}_d(\ell, a)$ is the cardinality of its label.

Note 2.3. (GENERATORS OF ISOCANTED CUBE) The vertices $\underline{1}, \underline{2}, \ldots, \underline{d}$ *and* $\underline{0}$ are called the *generators* of $\mathcal{I}_d(\ell, a)$. The rest of vertices are called *generated vertices* of $\mathcal{I}_d(\ell, a)$.

The following properties have been proved in [33]:

1. *Parent–child principle.* Two vertices in $\mathcal{I}_d(\ell, a)$ are joined by an edge if and only if they are labeled W and $W' \subsetneq [d] \cup \{0\}$ with $\emptyset \neq W \subsetneq W'$ and $|W| + 1 = |W'|$. We say that W is a *parent* of W' and that W' is a *child* of W.

2. 2*–face principle.* A 2–face of $\mathcal{I}_d(\ell, a)$ is determined by four vertices with labels $\{j\} \cup W$, $\{j, k\} \cup W$, $\{j, r\} \cup W$ and $\{j, k, r\} \cup W$, with $W \subseteq [d] \cup \{0\} \setminus \{j, k, r\}$, for $j, k, r \in [d] \cup \{0\}$ pairwise different.

Next we prove a generalization of the Items above which was assumed to be true in [33].

PROPOSITION 2.4. (s–FACE PRINCIPLE AND LABELING) *For each* $0 \leq s \leq d - 1$, *an* s*–face of* $\mathcal{I}_d(\ell, a)$ *is determined by two proper subsets* W *and* $W' \subsetneq [d] \cup \{0\}$ *with* $W \subseteq W'$ *and* $|W| + s = |W'|$*. It will be denoted* $F_{W,W'}$.

Proof. To $W'' \subseteq W' \setminus W$ it corresponds the vertex of $\mathcal{I}_d(\ell, a)$ labeled $\underline{W \cup W''}$ and we map this vertex to the s–tuple $\sum_{i \in W''} e_i$. In particular, to $\underline{W}$ we associate the zero s–tuple and to $\underline{W'}$ we associate the all–ones s–tuple. Based on the *parent–child principle* above, we get that a combinatorial s–cube $\mathcal{K}_s$ is combinatorially equivalent to the face $F_{W,W'}$ of $\mathcal{I}_d(\ell, a)$. ∎

In Figure 3 the labelling is shown for $d = 2$. The following result appears in [33] with an incomplete proof. It is a direct consequence of Proposition 2.4.

COROLLARY 2.5. (LATTICE ISOMORPHISM) *The lattice of proper faces of* $\mathcal{I}_d(\ell, a)$ *is isomorphic to the lattice of proper subsets of* $[d] \cup \{0\}$.

The following properties have been proved in [33, 34].

1. There is a unique combinatorial type for isocanted d–cubes.

2. $\mathcal{I}_d(\ell, a)$ is a cubical polytope. It is *almost simple*, for $d \geq 2$ (i.e., each of its vertices has valency either d or $d + 1$).

3. *Rotation invariance.* $\mathcal{I}_d(\ell, a)$ is invariant under the rotation in $\mathbb{R}^d$ with axis spanned by the all ones–vector u_d. and rotation angle $\frac{2\pi}{d}$. This axis joins the North and South poles of $\mathcal{I}_d(\ell, a)$. The matrix representing this rotation (with respect to the standard vector basis) is the *circulant matrix*

$$B_d = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{2.11}$$

Notice that B_d maps the generator $\underline{j}$ to the generator $\underline{j+1}$, for $j \in [d-1]$, and the generator $\underline{d}$ to the generator $\underline{1}$. Further, B_d fixes the generator $\underline{0} = \mathbf{S}$.[6]

4.
$$\mathrm{vol}_d\, \mathcal{I}_d(\ell, a) = \det(M_d(\ell, a)) = (\ell - a)^d + da\, (\ell - a)^{d-1}\,.$$

2.6. On incidence numbers. The *valency of a vertex* F in a d–polytope P is a well–known combinatorial notion depending only on the 1–skeleton of P. It is the number of edges of P meeting F. The *incidence numbers* introduced by Grünbaum are a generalization both of valency and of f–vector (cf. [23]).

Definition 2.6. (Grünbaum's incidence numbers) Let P be a d–polytope (or a $(d-1)$–polytopal complex) and F_j be a j–face of P, with $0 \leq j \leq d-1$.

For $0 \leq k \leq j$, define $h_k(F_j, P)$ to be the number of k–faces of P contained in F_j.

For $j \leq k \leq d-1$, define $h_k(F_j, P)$ to be the number of k–faces of P containing F_j.

In particular, $h_k(F_j, P) = f_k(F_j)$, the f–vector of the polytope F_j, for $0 \leq k \leq j$. Clearly, $h_1(F_0, P)$ is the valency of F in P and $h_j(F_j, P) = 1$. If P is a d–polytope, then $h_{d-1}(F_{d-2}, P) = 2$, i.e., in a d–polytope, each $(d-2)$–face is contained in exactly two facets (cf. [23, Theorem 3.1.6]).

[6]This rotation transforms generated vertices into generated vertices in a way one can easily describe using the labels.

Definition 2.7. (Face frequencies) Let P be a d–polytope (or a $(d-1)$–polytopal complex) and F be a face of P. The *metric frequency* of F in P is the number of faces of P congruent to F. It is denoted $mfr(F,P)$.[7]

Example 2.8.

1. $h_{j+1}(F_j, \Delta_d) = d-j$, because given a j–face F_j in Δ_d, any vertex of Δ_d not in F_j provides a $(j+1)$–face of Δ_d meeting F_j.
2. $h_{j+1}(F_j, \mathcal{C}_d) = d-j$, because given a j–face F_j in $\mathcal{C}_d$, any coordinate direction not in F_j provides a $(j+1)$–face of $\mathcal{C}_d$ meeting F_j.
3. A *rhombic dodecahedron* has 8 vertices with valency three and 6 vertices with valency four. In a cuboctahedron (which is dual of a rhombic dodecahedron), the combinatorial frequency of a triangular face is 8, and the combinatorial frequency of a quadrilateral face is 6. Incidentally, $\mathcal{I}_3(\ell,a)$ is combinatorially equivalent to a *rhombic dodecahedron*.

3. The j–volume of an isocanted cube

3.1. Isocanted cubes viewed as cubical zonotopes: the expression of vertices in two ways. In Subsection 2.5 and Subsection 2.4, we have seen two ways to express the vertices of $\mathcal{I}_d(\ell,a)$. For example, the *North Pole*

$$\mathbf{N} := \frac{\ell}{2}u_d = \underline{[d]} \ \text{ is written also as } \ y_1+y_2+\cdots+y_d+y_0, \tag{3.1}$$

where the y_j are given in (2.7). We set

$$\mathbf{I} := y_1+y_2+\cdots+y_d-y_0 = \frac{\ell-2a}{2}u_d. \tag{3.2}$$

Notice $\mathbf{I} \neq -\mathbf{I}$ if and only if $\ell \neq 2a$.

Lemma 3.1. (Interior points of isocanted cube)

$$i(\mathcal{I}_d(\ell,a)) = \{\pm\mathbf{I}\}. \tag{3.3}$$

Proof. Assume $\ell \neq 2a$. The point $\mathbf{I} = \frac{\ell-2a}{2}(e_1+e_2+\cdots+e_d)$ satisfies inequalities (1.1), since $0<a<\ell$ implies $-\frac{\ell}{2} \leq \frac{\ell-2a}{2} \leq \frac{\ell}{2}$. This shows that $\mathbf{I}$ belongs to $\mathcal{I}_d(\ell,a)$.

[7] A combinatorial frequency can also be considered.

Consider $\epsilon > 0$ and a unitary vector $v \in \mathbb{R}^d$. If we choose $\epsilon \le \min\{\frac{\ell-a}{2}, a\}$ we get that the point $\mathbf{I} + \epsilon v$ satisfies inequalities (1.1). This shows that $\mathbf{I}$ belongs to the interior of $\mathcal{I}_d(\ell, a)$ and therefore to $i(\mathcal{I}_d(\ell, a))$.

By symmetry, we obtain that $-\mathbf{I}$ also belongs to $i(\mathcal{I}_d(\ell, a))$, proving that the right hand side of (3.3) contained in the left hand side. The hypothesis $\ell \ne 2a$ yields $\mathbf{I} \ne -\mathbf{I}$. Now, using (2.4) and (2.5), we get

$$v(\mathcal{I}_d(\ell, a)) \subseteq \{\pm y_1 \pm y_2 \pm \cdots \pm y_d \pm y_0\} \setminus \{\pm \mathbf{I}\} \tag{3.4}$$

and, since the number of vertices of $\mathcal{I}_d(\ell, a)$ is $2^{d+1} - 2$, then we have equality in (3.4) and also in (3.3). If $\ell = 2a$, then $\mathbf{I} = -\mathbf{I}$, and the equalities (3.3) and (3.4) hold by continuous deformation. ∎

The proof of the following Lemma is a straight calculation.

Lemma 3.2. (Two expressions of the vertices of isocanted cube) *For each proper subset $W \subsetneq [d] \cup \{0\}$, we have*

$$\underline{W} = \begin{cases} \left(\sum_{j\in W} y_j\right) + y_0 - \left(\sum_{j\in[d]\setminus W} y_j\right) & \text{if } 0 \notin W, \\ \left(\sum_{j\in W} y_j\right) - y_0 - \left(\sum_{j\in[d]\setminus W} y_j\right) & \text{if } 0 \in W. \end{cases}$$

3.2. Hemispheres, supports and lengths of labels.

Definition 3.3. (Hemispheres) We say that the vertex $\underline{W}$ of $\mathcal{I}_d(\ell, a)$ lies in the *Northern hemisphere* if $0 \notin W$, and that lies in the *Southern hemisphere*, otherwise.

Remark 3.4. (Vertices are binary) The vertices of $\mathcal{I}_d(\ell, a)$ are *binary*. Those in the Northern (resp. Southern) hemisphere have coordinates in $\{\frac{\ell}{2}, \frac{2a-\ell}{2}\}$ (resp. $-\{\frac{\ell}{2}, \frac{2a-\ell}{2}\}$). This follows from (2.9) and (2.10).

Now we describe the faces of $\mathcal{I}_d(\ell, a)$. Recall that

$$\begin{aligned} &S_{\sigma(0)} + S_{\sigma(1)} + \cdots + S_{\sigma(d')} \\ +&\epsilon_{\sigma(d'+1)} y_{\sigma(d'+1)} + \epsilon_{\sigma(d'+2)} y_{\sigma(d'+2)} + \cdots + \epsilon_{\sigma(d)} y_{\sigma(d)} \end{aligned}$$

is a d'–dimensional *cell* in $\mathcal{I}_d(\ell, a)$, where σ is a permutation of the set $[d] \cup \{0\}$, $\epsilon_j = \pm 1$ and $0 \le d' \le d$. A face of $\mathcal{I}_d(\ell, a)$ is a cell C such that $v(C) \subseteq v(\mathcal{I}_d(\ell, a))$, by Lemma 2.1.

NOTATION 3.5. (SUPPORT OF A FACE OF AN ISOCANTED CUBE) Let F be a face of $\mathcal{I}_d(\ell, a)$. By $\operatorname{supp}(F) \subseteq [d] \cup \{0\}$, the *support* of F, we mean the union of the labels of the vertices of F.

For instance, $\operatorname{supp}(\operatorname{conv}(\underline{134}, \underline{1347}))) = \{1, 3, 4, 7\}$, with $d \geq 7$. In particular, a *generator* is a vertex whose support is a singleton (cf. Notation 2.3). A *d–generated vertex* is one whose support has cardinality d, i.e., it is the opposite of a generator, by the bijection $\mathcal{L}$ in (2.10).

Remark 3.6. The support function is *increasing*: If $F \subseteq F'$ are faces, then $\operatorname{supp}(F) \subseteq \operatorname{supp}(F')$, by the *s–face principle* (cf. Proposition 2.4).

NOTATION 3.7. (MAX, MIN, LMAX AND LMIN OF A FACE OF AN ISOCANTED CUBE) Let F be a face of $\mathcal{I}_d(\ell, a)$. By $\max(F)$ (resp. $\min(F)$) we denote the vertex of $\mathcal{I}_d(\ell, a)$ with maximal (resp. minimal) length (cf. Definition 2.2). It is unique, by the *s–face principle* (cf. Proposition 2.4).

By $\operatorname{lmax}(F)$ (resp. $\operatorname{lmin}(F)$) we denote the length of $\max(F)$ (resp. $\min(F)$). It holds

$$1 \leq \operatorname{lmin}(F) \leq \operatorname{lmax}(F) \leq d \text{ and } \operatorname{lmin}(F) + \dim(F) = \operatorname{lmax}(F), \tag{3.5}$$

by the *s–face principle*, (cf. Proposition 2.4). Further, lmin is decreasing and lmax in increasing:

$$\operatorname{lmin}(F) \geq \operatorname{lmin}(F'), \text{ and } \operatorname{lmax}(F) \leq \operatorname{lmax}(F'), \text{ for faces } F \subseteq F'. \tag{3.6}$$

For example, $\max(\mathcal{I}_d(\ell, a)) = \underline{[d]} = \mathbf{N}$, $\operatorname{lmax} \mathcal{I}_d(\ell, a) = d$, $\min(\mathcal{I}_d(\ell, a)) = \underline{0} = \mathbf{S}$ and $\operatorname{lmin} \mathcal{I}_d(\ell, a) = 1$.

The next Remark is immediate to check.

Remark 3.8. (MAX–MIN PRINCIPLE FO R AN ISOCANTED CUBE) With notations from the *s–face principle* (cf. Proposition 2.4), we have

$$\min(F_{W,W'}) = \underline{W}, \qquad \max(F_{W,W'}) = \underline{W'}, \qquad \operatorname{lmin}(F_{W,W'}) = |W|,$$
$$\operatorname{lmax}(F_{W,W'}) = |W'|, \qquad \dim(F_{W,W'}) = |W'| - |W|.$$

3.3. TWO TYPES OF FACES IN AN ISOCANTED CUBE. Warning: in this section Polar means "touching one Pole".

LEMMA 3.9. (EXCLUSION LEMMA) *Each facet of $\mathcal{I}_d(\ell, a)$ contains one Pole or contains the component $S_0 = \operatorname{conv}(-y_0, y_0)$, but not both.*

Proof. A facet of $\mathcal{I}_d(\ell, a)$ is a cell C of dimension $d-1$ not containing interior points of $\mathcal{I}_d(\ell, a)$. Recall the points y_j defined in (2.7). Since each subset of $\{y_1, y_2, \dots, y_d, y_0\}$ of cardinality $d-1$ is linearly independent, we have three types of $(d-1)$–cells which, up to a permutation of indices and a change of signs, can be expressed as follows: $C = S_1 + \cdots + S_{d-1} + y_d + y_0$, $C = S_1 + \cdots + S_{d-1} + y_d - y_0$, or $C = \pm y_1 \pm y_2 + S_3 + \cdots + S_d + S_0$.

If $C = S_1 + \cdots + S_{d-1} + y_d + y_0$, then the North Pole $\mathbf{N}$ lies in C, by (3.1). If $C = S_1 + \cdots + S_{d-1} + y_d - y_0$, then the interior point $\mathbf{I}$ lies in C, by (3.2), so C is not a facet of $\mathcal{I}_d(\ell, a)$. Finally, if $C = \pm y_1 \pm y_2 + S_3 + \cdots + S_d + S_0$, then S_0 is a component of C. ∎

Definition 3.10. (Polar and Equatorial faces) A face F of $\mathcal{I}_d(\ell, a)$ is called *Polar* if it meets one Pole, and it is called *Equatorial* if S_0 is a component of F, i.e., if F belongs to the 0–th zone.

By the *Exclusion Lemma* (cf. Lemma 3.9), a facet is either Polar or Equatorial, but not both. A j–face can be neither Polar nor Equatorial, for $1 \le j \le d-2$.

Proposition 3.11. *For each Polar j–face F of $\mathcal{I}_d(\ell, a)$ we have* $\mathrm{vol}_j(F) = (\ell - a)^j$.

Proof. A Polar face is congruent to a cube. ∎

Lemma 3.12. (Distance lemma) *For $1 \le j \le d-1$, let F be a j–face of $\mathcal{I}_d(\ell, a)$ touching the North Pole. Let $\underline{W}$ be any vertex of F different from $\mathbf{N}$. Then*

$$\mathrm{d}\left(\underline{W \cup \{0\}}\,, \mathrm{aff}(F)\right) = a\sqrt{d-j}\,,$$

and it depends neither on ℓ nor on the choice of $\underline{W}$.

Proof. Let us assume, without loss of generality, that $\mathrm{aff}(F) : x_{j+1} = x_{j+2} = \cdots = x_d = \frac{\ell}{2}$.

For $1 \le j \le d-1$, let Y be the point where the affine subspace passing through $\underline{W \cup \{0\}}$ and orthogonal to $\mathrm{aff}(F)$ meets $\mathrm{aff}(F)$. Then $\underline{W}$, $\underline{W \cup \{0\}}$ and Y determine a right triangle, whose hypothenuse is

$$E := \mathrm{conv}\left(\underline{W \cup \{0\}}\,, \underline{W}\right),$$

a segment parallel to the all–ones vector u_d and congruent to S_0. The length of E is $a\sqrt{d}$. One leg of the triangle, of length $a\sqrt{j}$, is contained in $\mathrm{aff}(F)$.

The distance we want to compute is the other leg, and we apply Pythagoras theorem. ∎

PROPOSITION 3.13. *For each* $0 \le j \le d-2$ *and each Equatorial* $(j+1)$–*face* F' *in* $\mathcal{I}_d(\ell, a)$, *we have* $\operatorname{vol}_j(F') = (\ell - a)^j a\sqrt{d-j}$.

Proof. We have $F' = S_0 + F$, where F is a Polar j–face of $\mathcal{I}_d(\ell, a)$. Further, F' is a parallelepiped with basis F of area $(\ell - a)^j$, by Proposition 3.11 and height given by Lemma 3.12. ∎

We can split the f–vector of $\mathcal{I}_d(\ell, a)$ into the Polar part and the Equatorial part, due to the Exclusion Lemma (cf. Lemma 3.9). Denote

$$f_{j,P}(\mathcal{I}_d) := 2\left(2^{d-j} - 1\right)\binom{d}{j} = \left(2^{d-j+1} - 2\right)\binom{d}{j} \tag{3.7}$$

which is twice the f–vector of a vertex cask $\mathcal{V}_{d-1}$ (cf. expression (2.2)) and let

$$f_{j,E}(\mathcal{I}_d) := f_j(\mathcal{I}_d) - f_{j,P}(\mathcal{I}_d).$$

Notice

$$f_{j,E}(\mathcal{I}_d) = (2^{d-j+1} - 2)\binom{d}{j-1} = f_{j-1}(\mathcal{I}_{d-1}). \tag{3.8}$$

In particular, the Equatorial part of the f–vector counts no vertices.

THEOREM 3.14. *For each dimension* $d \ge 2$, *real numbers* $0 < a < \ell$ *and* $0 \le j \le d-1$, *we have*

$$\begin{aligned} \operatorname{vol}_j(\mathcal{I}_d(\ell, a)) = 2\left(2^{d-j} - 1\right)\binom{d}{j}(\ell - a)^j \\ + (2^{d+1-j} - 2)\binom{d}{j-1}(\ell - a)^{j-1} a\sqrt{d-j+1} \end{aligned} \tag{3.9}$$

is a polynomial in ℓ, a *homogeneous of degree* j, *with coefficients in* $\mathbb{Q}[\sqrt{d-j+1}]$.

Proof. Using the *Exclusion Lemma*, Propositions 3.11 and 3.13, and the expressions (3.7) and (3.8), we get

$$\operatorname{vol}_j(\mathcal{I}_d(\ell, a)) = f_{j,P}(\mathcal{I}_d)(\ell - a)^j + f_{j,E}(\mathcal{I}_d)(\ell - a)^{j-1} a\sqrt{d-j+1}.$$

∎

Remark 3.15. Expression (3.9) is not valid for the limit cases $j = d$, $a = 0, \ell$. Indeed, $\mathrm{vol}_d(\mathcal{I}_d(\ell, \ell)) = 0$, because the dimension of $\mathcal{I}_d(\ell, \ell)$ is $d - 1$. If $a = 0$ then $\mathcal{I}_d(\ell, 0)$ is a d–cube of edge–length ℓ.

EXAMPLE 3.16. If $d = 3$, $f_P(\mathcal{I}_3) = 2(7, 9, 3)$, $f_E(\mathcal{I}_3) = (0, 6, 6)$. From (3.9)

$$\mathrm{vol}_1(\mathcal{I}_3(\ell, a)) = 18(\ell - a) + 6a\sqrt{3 - 1 + 1} = 18(\ell - a) + 6a\sqrt{3},$$
$$\mathrm{vol}_2(\mathcal{I}_3(\ell, a)) = 6(\ell - a)^2 + 6(\ell - a)a\sqrt{3 - 2 + 1} = 6(\ell - a)^2 + 6(\ell - a)a\sqrt{2}.$$

4. THE ISOCANTED CUBE IS A TILE

We know that $\mathcal{I}_d(\ell, a)$ is the zonotope arising from the configuration of vectors $Y := (y_1, y_2, \dots, y_d, y_0)$ given in (2.7). Recall that the labelling of facets is given by the *s–face principle* in Proposition 2.4. How does this labeling show that two facets are opposite? For fixed $j \in [d]$, the facet $F_{j,[d]}$ is supported by the hyperplane of equation $x_j = \frac{\ell}{2}$. Similarly, for fixed $i \neq j \in [d]$, the facet $F_{i,[d]\cup\{0\}\setminus\{j\}}$ is supported by the hyperplane of equation $x_i - x_j = \ell - a$. By the bijection $\mathcal{L}$ in (2.10), the facet contained in $-x_j = \frac{\ell}{2}$ is $F_{0,[d]\cup\{0\}\setminus\{j\}}$, and the facet contained in $-x_i + x_j = \ell - a$ is $F_{j,[d]\cup\{0\}\setminus\{i\}}$. In symbols:

$$-F_{j,[d]} = F_{0,[d]\cup\{0\}\setminus\{j\}}, \quad -F_{i,[d]\cup\{0\}\setminus\{j\}} = F_{j,[d]\cup\{0\}\setminus\{i\}}, \quad i \neq j \in [d].$$

Next, the following two Lemmas show that the barycenters of the facets of $\mathcal{I}_d(\ell, a)$ arise from the columns of the Bose–Mesner matrix

$$M_d(\ell, a) = \begin{pmatrix} \ell & a & \cdots & a \\ a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a \\ a & \cdots & a & \ell \end{pmatrix}. \tag{4.1}$$

LEMMA 4.1. (BARYCENTER OF FACET $F_{j,[d]}$) *For $j \in [d]$, the barycenter of the facet $F_{j,[d]}$ is* HALF *the j–th column of $M_d(\ell, a)$.*

Proof. Without loss of generality, we can assume $j = d$ (by rotation in Item 3 in p. 10). The labels of the vertices of $F_{d,[d]}$ are those subsets of $[d]$ which contain d, and thus we have $|v(F_{d,[d]})| = 2^{d-1}$.

Given $d \in W \subseteq [d]$, the coordinates of the vertex $\underline{W}$ are maximal and equal to $\frac{\ell}{2}$, for an index belonging to W, and minimal and equal to $\frac{2a-\ell}{2}$, for an index belonging to $[d] \setminus W$. The baricenter of $F_{d,[d]}$ is

$$\frac{1}{2^{d-1}} \sum_{P \in v(F_{d,[d]})} P = \sum_{k=1}^{d} \lambda_k e_k.$$

The contribution to e_d is constant and equal to $\frac{\ell}{2}$, for each $P \in v(F_{d,[d]})$, and therefore $\lambda_d = \frac{\ell}{2}$. For $k \neq d$, the contribution to e_k is equal to $\frac{\ell}{2}$, for half of the vertices, and equal to $\frac{2a-\ell}{2}$, for the other half, and therefore $\lambda_k = \frac{a}{2}$, proving the result. ∎

LEMMA 4.2. (BARYCENTER OF FACET $F_{i,[d]\cup\{0\}\setminus\{j\}}$) *For $i \neq j \in [d]$, the barycenter of the facet $F_{i,[d]\cup\{0\}\setminus\{j\}}$ is* HALF *of the difference of columns i–th and j–th of $M_d(\ell, a)$.*

Proof. Without loss of generality, we can assume $j = d$ (by rotation 3). Given W such that $i \in W \subseteq [d-1] \cup \{0\}$, we have two cases. If $0 \notin W$, then the coordinates of the vertex $\underline{W}$ are maximal and equal to $\frac{\ell}{2}$, for an index belonging to W, and minimal and equal to $\frac{2a-\ell}{2}$, for an index belonging to $[d-1] \setminus W$. However, if $0 \in W$, then the coordinates of the vertex $\underline{W}$ are maximal and equal to $\frac{\ell-2a}{2}$, for an index belonging to $W \setminus \{0\}$, and minimal and equal to $\frac{-\ell}{2}$, for an index belonging to $[d-1] \setminus W$. The barycenter of this facet is

$$\frac{1}{2^{d-1}} \sum_{P \in v(F_{i,[d-1]\cup\{0\}})} P = \sum_{k=1}^{d} (\mu_k + \lambda_k) e_k$$

where μ_k is the contribution of those vertices $P = \underline{W}$ with $0 \notin W$, and λ_k is the contribution of vertices $P = \underline{W}$ with $0 \in W$.

The contribution to e_i is constant and equal to $\frac{\ell}{2}$, for each $P = \underline{W}$ with $0 \notin W$, so $\mu_i = \frac{1}{2^{d-1}} \frac{\ell}{2} 2^{d-2} = \frac{\ell}{4}$, and it is constant and equal to $\frac{\ell-2a}{2}$, for each $P = \underline{W}$ with $0 \in W$, so $\lambda_i = \frac{1}{2^{d-1}} \frac{\ell-2a}{2} 2^{d-2} = \frac{\ell-2a}{4}$, and therefore $\mu_i + \lambda_i = \frac{\ell-a}{2}$.

The contribution to e_d is constant and equal to $\frac{2a-\ell}{2}$, for each $P = \underline{W}$ with $0 \notin W$, so $\mu_d = \frac{1}{2^{d-1}} \frac{2a-\ell}{2} 2^{d-2} = \frac{2a-\ell}{4}$, and it is constant and equal to $\frac{-\ell}{2}$, for each $P = \underline{W}$ with $0 \in W$, so $\lambda_d = \frac{1}{2^{d-1}} \frac{-\ell}{2} 2^{d-2} = \frac{-\ell}{4}$, and therefore $\mu_d + \lambda_d = \frac{a-\ell}{2}$.

For each k with $i \neq k \neq d$, the contribution to e_k is zero, because one fourth of the vertices contribute $\frac{\ell-2a}{2}$, another fourth of the vertices contribute $\frac{2a-\ell}{2}$, another fourth of the vertices contribute $\frac{a}{2}$, and another fourth of the vertices contribute $\frac{-a}{2}$. The result is proved. ∎

Following McMullen's paper [29], we consider the lattice Λ spanned by the barycenters of the facets of $\mathcal{I}_d(\ell, a)$, each one MULTIPLIED BY TWO. By Lemmas 4.1 and 4.2, Λ is spanned by the columns of the Bose–Mesner matrix $M_d(\ell, a)$.

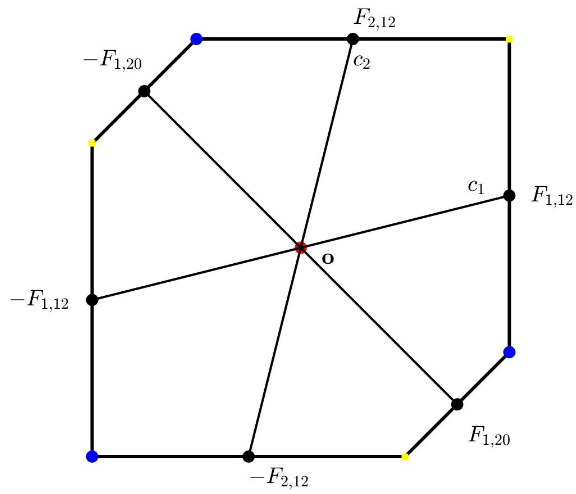

Figure 3: Isocanted square ($d = 2$) with labeling of facets and barycenters of facets. The origin marked in red.

THEOREM 4.3. (TILE SPACE) *$\mathcal{I}_d(\ell, a)$ tiles $\mathbb{R}^d$ face–to–face by translations.*

Proof. Consider the matrix E of size $(d+1) \times d(d+1)$ over $\{-1, 0, 1\}$, whose columns express the barycenters of the facets of $\mathcal{I}_d(\ell, a)$ as linear combinations of the vectors $y_1, y_2, \ldots, y_d, y_0$. Recall $y_j = \frac{\ell-a}{2} e_j$, $j \in [d]$ and $y_0 = \frac{a}{2} u_d = \frac{a}{2}(e_1 + e_2 + \ldots + e_d)$. It is enough to show that $\operatorname{rk} E = d$, by condition III in [29].

The columns corresponding to the barycenters of the facets $F_{1,[d]}, F_{2,[d]}, \ldots, F_{d,[d]}$ are gathered in the following matrix E_1 (which is a submatrix of E):

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix} \in M_{(d+1)\times d}$$

because the barycenter c_j of $F_{j,[d]}$ satisfies $c_j = y_j + y_0$, by Lemma 4.1. Clearly, $d = \operatorname{rk} E_1 \leq \operatorname{rk} E \leq d$, and thus condition III has been proved. ∎

Remark 4.4. Here is an alternative proof of Theorem 4.3. The lattice Λ is spanned by the columns of the Bose–Mesner matrix, by Lemmas 4.1 and 4.2. The determinant of Λ (i.e., the determinant of a *fundamental parallelepiped* of Λ) is $\det(\Lambda) = \det(M_d(\ell, a))$. Following [46], $\mathcal{I}_d(\ell, a) + \Lambda$ is a lattice packing of $\mathbb{R}^d$, whose density is

$$\frac{\operatorname{vol}_d(\mathcal{I}_d(\ell, a))}{\det(\Lambda)} = \frac{\det(M_d(\ell, a))}{\det(M_d(\ell, a))} = 1.$$

This proves that the packing is actually a tiling and $\mathcal{I}_d(\ell, a)$ is a tile.

5. The symmetry group of $\mathcal{I}_d(\ell, a)$

Let P be a polytope. A *symmetry* of P is an isometry of $\operatorname{aff}(P)$ that maps P to itself. The *symmetry group* of P is the group $G(P)$ of all the symmetries of P. The symmetry group of polytopes is studied in [35, 37].

Let C_d denote the cyclic group of order d.

Proposition 5.1. *The symmetry group of $\mathcal{I}_d(\ell, a)$ contains the group $C_d \times C_2$ as a subgroup.*

Proof. Let $\rho : \mathbb{R}^d \to \mathbb{R}^d$ be the rotation given by the matrix B_d in (2.11). Then ρ generates a cyclic group of order d. In addition, $-\operatorname{id}$ maps $\mathcal{I}_d(\ell, a)$ to itself since, restricted to $v(\mathcal{I}_d(\ell, a))$, it maps $\underline{W}$ to $\underline{[d] \cup \{0\} \setminus W}$. On can prove that ρ and $-\operatorname{id}$ commute, whence $C_d \times C_2$ is a subgroup of $G(\mathcal{I}_d(\ell, a))$. ∎

6. Incidence numbers for isocanted cubes

Recall the incidence numbers introduced in Definition 2.6 and the minimal length of a face introduced in Notation 3.7.

Proposition 6.1. (Incidence numbers for isocanted cubes) *For $0 \leq s \leq d-1$, let F_s be an s–face of $\mathcal{I}_d(\ell, a)$ with $w = \operatorname{lmin}(F_s)$. Then for $s \leq k \leq d-1$ we have*

$$h_k(F_s, \mathcal{I}_d(\ell, a)) = \sum_{r=\max\{0,k+w-d\}}^{\min\{w-1,k-s\}} \binom{w}{r} \binom{d+1-w-s}{k-s-r}. \tag{6.1}$$

Proof. Fix s and k.

If $w = 1$ then $\min(F_s)$ is a generator (i.e., F_s has no parents), whence $1 = \operatorname{lmin}(F_s) = \operatorname{lmin}(F_k)$, by (3.6) and so $\min(F_s) = \min(F_k)$, for each k–face F_k containing F_s. In order to determine F_k completely, we must choose, in all possible ways, $r = k - s$ elements in the set $[d] \cup \{0\} \setminus \operatorname{supp}(\max(F_s))$, where $|[d]\cup\{0\}\setminus\operatorname{supp}(\max(F_s))| = d+1-\operatorname{lmax}(F_s) = d+1-(1+s) = d-s$, by (3.5). The union of the $k-s$ elements and $\operatorname{supp}(\max(F_s))$ is exactly $\operatorname{supp}(\max(F_k))$, with $\operatorname{lmax}(F_k) = 1+k$, by the *max–min principle*, (cf. Remark 3.8). Thus, in this case, $h_k(F_s) = \binom{d-s}{k-s}$. Looking at the right–hand–side of (6.1), the lower limit is $\max\{0, k+1-d\} = 0$, the upper limit $\min\{1-1, k-s\} = 0$, so $r = 0$ yields the only term in the sum: $\binom{w}{r}\binom{d+1-w-s}{k-s-r} = \binom{1}{0}\binom{d+1-1-s}{k-s-0} = 1\binom{d-s}{k-s}$, proving the equality.

If $w = d$ then $\max(F_s)$ is a d–generated vertex (i.e., F_s has no children) and the proof goes similarly.

The general case is $1 < w < d-1$. We have $\operatorname{lmax}(F_s) = w + s \leq d$, by (3.5). Let F_k be a k–face containing F_s. We have $1 \leq \operatorname{lmin}(F_k) \leq \operatorname{lmin}(F_s)$ and the second inequality may be strict. In order to determine F_k completely, we must erase r elements from $\operatorname{supp}(\min(F_s))$ (to produce $\min(F_k)$) and add $k-s-r$ elements taken from $[d] \cup \{0\} \setminus \operatorname{supp}(\max(F_s))$ to $\operatorname{supp}(\max(F_s))$ (to produce $\max(F_k)$), in all possible ways. Since $\operatorname{lmax}(F_s) = w + s$, this yields

$$h_k(F_s, \mathcal{I}_d(\ell, a)) = \sum_r \binom{w}{r} \binom{d+1-w-s}{k-s-r}.$$

The limits for the variation of the index r are given by $0 \leq r < w$, $0 \leq r \leq k-s$ and $0 \leq k-s-r < d+1-s-w$. Then the equality (6.1) follows. ∎

Let us check some special cases of expression (6.1).

- If $s=k$, then $k+w=\dim(F_k)+\operatorname{lmin}(F_k)=\operatorname{lmax}(F_k)\leq d$, by (3.5), whence

$$h_s(F_s,\mathcal{I}_d(\ell,a))=\sum_{r=0}^{0}\binom{w}{0}\binom{d+1-w-s}{0-0}=1,$$

which is correct.

- If $s=0$ and $k=1$, then

$$h_1(F_0,\mathcal{I}_d(\ell,a))=\sum_{r=\max\{0,1+w-d\}}^{\min\{w-1,1-0\}}\binom{w}{r}\binom{d+1-w-0}{1-0-r}$$

$$=\begin{cases}\binom{1}{0}\binom{d+1-1}{1-0}=d & \text{if } \ w=1,\\ \binom{d}{1}\binom{d+1-d}{1-1}=d & \text{if } \ w=d,\\ \binom{w}{0}\binom{d+1-w}{1-0}+\binom{w}{1}\binom{d+1-w}{1-1}=d+1 & \text{else,}\end{cases}$$

which is equal to the valency of a vertex F_0 in $\mathcal{I}_d(\ell,a)$ and is correct (because $\mathcal{I}_d(\ell,a)$ is *almost simple*; cf. Item 2 in the list in p. 10).

- If $s=d-2$ and $k=d-1$, then we are counting in how many facets a $(d-2)$–face is contained —the obvious answer is two, for any d–polytope. We have

$$h_{d-1}(F_{d-2},\mathcal{I}_d(\ell,a))=\sum_{r=w-1}^{\min\{w-1,1\}}\binom{w}{r}\binom{d+1-w-s}{k-s-r}=2.$$

Indeed, if $w=1$, then we get

$$h_{d-1}(F_{d-2},\mathcal{I}_d(\ell,a))=\sum_{r=0}^{0}\binom{1}{r}\binom{d+1-1-d+2}{1-r}=\binom{1}{0}\binom{2}{1-0}=2.$$

If $w=2$, then we get

$$h_{d-1}(F_{d-2},\mathcal{I}_d(\ell,a))=\sum_{r=1}^{1}\binom{2}{r}\binom{d+1-2-d+2}{1-r}=\binom{2}{1}\binom{1}{0}=2.$$

Now, $w>2$ is impossible for a $(d-2)$–face because, by (3.5), it holds $w+(d-2)=\operatorname{lmin}(F_{d-2})+\dim(F_{d-2})=\operatorname{lmax}(F_{d-2})\leq d$.

COROLLARY 6.2. *For* $0 \le s \le d-1$ *let* F_s *be an* s*–face of* $\mathcal{I}_d(\ell, a)$ *with* $w = \operatorname{lmin}(F_s)$. *Then*

$$h_{d-1}(F_s, \mathcal{I}_d(\ell, a)) = \sum_{r=w-1}^{\min\{w-1, d-1-s\}} \binom{w}{r} \binom{d+1-w-s}{d-1-s-r}.$$

7. Questions and examples

1. Compute the ratio $\operatorname{vol}_{d-1}(\mathcal{I}_d(\ell, a)) / \operatorname{vol}_d(\mathcal{I}_d(\ell, a))$ and compare the result with other familiar convex bodies.

2. The *width* $w_u(K)$ of a d–dimensional convex body K in the direction of a given by a unit vector u is, by definition, the distance between the two parallel supporting hyperplanes of K which are orthogonal to u. The *mean width* of K is the average of all $w_u(K)$. In [27] one finds a formula for the mean width of a zonotope. Compute the mean width of $\mathcal{I}_d(\ell, a)$.

3. Compute both the lattice and the affine equivalence given by *Erdahl's Theorem* (cf. p. 2).

4. Can isocanted cubes be used as polytopal approximations of ellipsoids?

5. We know that if the polytope P is a lattice tile with lattice Λ generated by the columns of a matrix M, then $\operatorname{vol}(P) = \det(M)$. This holds for cubes and isocanted cubes. Find polytopes P for which there exist a function $f : \mathbb{R} \to \mathbb{R}$ and a matrix M such that $\operatorname{vol}(P) = f(\det(M))$? One example are d–simplices, taking $f(x) = x$, or taking $f(x) = \sqrt{x}$, by the *Cayley–Menger determinant* or the *Gram matrix*. (The references [15, 20] may be useful here.)

6. Is there a relation between isocanted cubes and tight frames? (cf. [25])

Acknowledgements

We thank P.L. Clavería for producing Figure 1.

References

[1] R.M. Adin, A new cubical h-vector, in "Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics" (New Brunswick, NJ, 1994; edited by R. Simion), *Discrete Math.* **157** (1-3) (1996), 3 – 14. doi.org/10.1016/S0012-365X(96)83003-2

[2] M. Alexander, M. Fradelizi, L.C. García-Lirola, A. Zvavitch, Geometry and volume product of finite dimensional Lipschitz–free spaces, *J. Funct. Anal.* **280** (2021), paper n. 108849, 38 pp. doi.org/10.1016/j.jfa.2020.108849

[3] A.D. Alexandrov, "Convex polyhedra", Springer-Verlag, Berlin, 2005.

[4] M. Beck, S. Robins, "Computing the continuous discretely. Integer–point enumeration in Polyhedra", Springer, New York, 2007.

[5] L.J. Billera, A. Björner, Face numbers of polytopes and complexes, Chapter 17 in [19] in this list.

[6] T. Bisztriczky, P. McMullen, R. Schneider, I. Weiss (eds.), "Polytopes: abstract, convex and computational", NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 440, Kluwer Academic Publishers Group, Dordrecht, 1994.

[7] G. Blind, R. Blind, The cubical d-polytopes with fewer than 2^{d+1} vertices, *Discrete Comput. Geom.* **13** (3-4) (1995), 321 – 345. doi.org/10.1007/BF02574048.

[8] G. Blind, R. Blind, The almost simple cubical polytopes, *Discrete Math.* **184** (1998), 25 – 48. doi.org/10.1016/s0012-365x(97)00159-3

[9] V. Boltyanski, H. Martini, P.S. Soltan, "Excursions into combinatorial geometry", Springer-Verlag, Berlin, 1997.

[10] R.C. Bose, A note on Fisher's inequality for balanced incomplete block designs, *Ann. Math. Statistics* **20** (1949), 619 – 620. doi.org/10.1214/aoms/1177729958.

[11] R.C. Bose, D.M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced dessigns, *Ann. Math. Statist.* **30** (1959), 21 – 38. www.jstor.org/stable/2237117.

[12] N. Carlini, D. Paleka, K. (Dj) Dvijotham, T. Steinke, J. Hayase, A. Feder Cooper, K. Lee, M. Jagielski, M. Nasr, A. Conmy, I. Yona, E. Wallace, D. Rolnick, F. Tramèr, Stealing part of a production language model, arXiv:2403.06634v2 (9 Jul 2024).

[13] H.M.S. Coxeter, The classification of zonohedra by means of projective diagrams, *J. Math. Pures Appl. (9)* **41** (1962), 137 – 156.

[14] P.R. Cromwell, "Polyhedra", Cambridge Univ. Press, Cambridge, 1997.

[15] A.M. Dall, "Matroids: h-vectors, zonotopes and Lawrence polytopes", Thesis, U. Politècnica de Catalunya, 2015.

[16] M. Deza, V. Grishukhin, Voronoi's conjecture and space tiling zonotopes, *Mathematika* **51** (2004), 1 – 10.

[17] R.M. Erdahl, Zonotopes, dicings, and Voronoi's conjecture on parallelohedra, *European J. Combin.* **20** (1999), 527 – 549.

[18] R.J. Gardner, "Geometric tomography", Encyclopedia Math. Appl., 58, Cambridge University Press, Cambridge, 1995.

[19] J.E. Goodman, J. O'Rourke, C.D. Tóth (eds.), "Handbook of discrete and computational geometry", Third edition, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2018.

[20] E. Gover, N. Krikorian, Determinants and the volumes of parallelotopes and zonotopes, *Linear Algebra Appl.* **433** (2010), 28 – 40. doi.org/10.1016/j.laa.2010.01.031.

[21] P. Gritzmann, V. Klee, Computational convexity, Chapter 36 in [19] in this list.

[22] P.M. Gruber, J.M. Wills (Eds.), "Convexity and its applications", Birkhäuser Verlag, Basel-Boston, Mass., 1983.

[23] B. Grünbaum, "Convex polytopes", Second edition, Grad. Texts in Math., 221, Springer-Verlag, New York, 2003.

[24] M. Henk, J. Richter–Gebert, G.M. Ziegler, Basic properties of convex polytopes, Chapter 15 in [19] in this list.

[25] G. Ivanov, Tight frames and related geometric problems, *Canad. Math. Bull.* **64** (4) (2021), 942 – 963.

[26] W. Jockusch, The lower and upper bound problems for cubical polytopes, *Discrete Comput. Geom.* **9** (2) (1993), 159 – 163. doi.org/10.1007/BF02189315.

[27] A. Joós, Z. Lángi, Isoperimetric problems for zonotopes, *Mathematika* **69** (2) (2023), 508 – 534.

[28] P. McMullen, On zonotopes, *Trans. Amer. Math. Soc.* **159** (1971), 91 – 109.

[29] P. McMullen, Space tiling zonotopes, *Mathematika* **22** (1975), 202 – 211.

[30] P. McMullen, Convex bodies which tile space by translation, *Mathematika* **27** (1980), 113 – 121.

[31] P. McMullen, Acknowledgement of priority: "Convex bodies which tile space by translation", *Mathematika* **28** (2) (1981), 191.

[32] P. Mürner, Translative Parkettierungspolyeder und Zerlegungsgleichheit, *Elem. Math.* **30** (1975), 25 – 27.

[33] M.J. de la Puente, P.L. Clavería, Isocanted alcoved polytopes, *Appl. Math.* **65** (6) (2020), 703 – 726.

[34] M.J. de la Puente, P.L. Clavería, The volume of an isocanted cube is a determinant, *Linear and Multilinear Algebra* (03 Jul 2024) doi.org/10.1080/03081087.2024.2368240.

[35] S.A. Robertson, Polytopes and symmetry, London Math. Soc. Lecture Note Ser., 90, Cambridge University Press, Cambridge, 1984.

[36] R. Schneider, "Convex bodies: the Brunn–Minkowski theory", Encyclopedia Math. Appl., 44, Cambridge University Press, Cambridge, 1993.

[37] E. Schulte, Symmetry of polytopes and polyhedra, Chapter 18 in [19] in this list.

[38] M. Senechal (ed.) "Shaping space", Exploring polyhedra in nature, art, and the geometrical imagination, Springer, New York, 2013.

[39] G.C. Shephard, Combinatorial properties of associated zonotopes, *Canadian J. Math.* **26** (1974), 302 – 321.

[40] V. Soltan, "Lectures on convex sets", Seocnd edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

[41] A.C. Thompson, "Minkowski geometry", Encyclopedia Math. Appl., 63, Cambridge University Press, Cambridge, 1996.

[42] F. Vallentin, A note on space tiling zonotopes, `arXiv:math/0402053`, 2004.

[43] R. Webster, "Convexity", Oxford Sci. Publ., The Clarendon Press, Oxford University Press, New York, 1994.

[44] G.M. Ziegler, "Lectures on polytopes", Grad. Texts in Math., 152, Springer-Verlag, New York, 1995.

[45] G.M. Ziegler, Convex polytopes: extremal constructions and f-vector shapes, in "Geometric combinatorics", IAS/Park City Math. Ser., 13, American Mathematical Society, Providence, RI, 2007, 617 – 691.

[46] C. Zong, "Strange phenomena in convex and discrete geometry", Universitext, Springer-Verlag, New York, 1996.

[47] C. Zong, What is known about unit cubes, *Bull. Amer. Math. Soc. (N.S.)* **42** (2) (2005), 181 – 211.