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A study on W_9-curvature tensor within the framework of Lorentzian para-Sasakian manifold

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Abstract: This article focuses on the study of Lorentzian para-Sasakian manifolds $\mathcal{M}^n$. It demonstrates that a W_9-semisymmetric Lorentzian para-Sasakian manifold is a W_9-flat manifold. Additionally, we explore Lorentzian para-Sasakian manifolds that satisfy the ζ-W_9-flat condition, revealing that they represent a special type of η-Einstein manifold. Furthermore, it is shown that a W_9-flat Lorentzian para-Sasakian manifold is a flat manifold. We also investigate Lorentzian para-Sasakian manifolds that meet W_9-recurrent and ϕ-W_9-semisymmetric conditions, presenting several significant results from this analysis. At last, we explore η-Ricci Solitons on Lorentzian para-Sasakian manifold satisfying $W_9(\zeta, \mathcal{F}_1) \cdot S = 0$.

Key words: W_9-curvature tensor, Lorentzian para-Sasakian manifolds, η-Einstein manifolds, η-Ricci solitons.

MSC (2020): 53C15, 53C25, 53D15.

1. Introduction

In 1989, Matsumoto [17] introduced the concept of Lorentzian para-Sasakian manifolds. Shortly thereafter, Mihai and Roșca [20] independently defined the same notion, contributing several important results regarding these manifolds. The study of Lorentzian para-Sasakian manifolds has continued to grow, with significant contributions from various researchers. Matsumoto and Mihai [18] collaborated on further investigations, while Matsumoto, Mihai, and Roşca [19] collectively explored additional aspects of these manifolds.

Further research was conducted by Mihai, Shaikh, and De [21], who examined specific properties and characteristics of Lorentzian para-Sasakian manifolds. Additionally, De and Shaikh [10, 9], along with Ozgur [24], also contributed to the body of knowledge surrounding this topic. Their collective efforts have enriched the understanding of Lorentzian para-Sasakian manifolds, leading to new insights and findings in differential geometry.

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The exploration of Lorentzian para-Sasakian manifolds has not only expanded theoretical frameworks but has also opened avenues for applications in various mathematical contexts. As a result, the study of these manifolds continues to attract interest from researchers, fostering ongoing investigation and discovery in the field. Overall, the foundational work laid by Matsumoto and subsequent contributions like [30, 15, 22, 32, 33] by other scholars has significantly advanced the study of Lorentzian para-Sasakian manifolds, establishing a vibrant area of research within differential geometry.

The τ curvature tensor is defined by Tripathi and Gupta [34] as

$$\begin{aligned}\tau(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = a_0R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 &+ a_1S(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 + a_2S(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2\\ &+ a_3S(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + a_4g(\mathcal{F}_2,\mathcal{F}_3)Q\mathcal{F}_1\\ &+ a_5g(\mathcal{F}_1,\mathcal{F}_3)Q\mathcal{F}_2 + a_6g(\mathcal{F}_1,\mathcal{F}_2)Q\mathcal{F}_3\\ &+ a_7r(g(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2)\end{aligned} \tag{1.1}$$

where $a_0,\dots,a_7$ are some smooth functions on $\mathcal{M}^n$; and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator of type (1,1) and the scalar curvature respectively.

Substituting $a_0 = 1$, $a_3 = -a_4 = \frac{1}{n-1}$, $a_1 = a_2 = a_5 = a_6 = a_7 = 0$ in (1.1), we obtain

$$W_9(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3+\frac{1}{(n-1)}[S(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3-g(\mathcal{F}_2,\mathcal{F}_3)Q\mathcal{F}_1] \tag{1.2}$$

where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator of type (1,1) respectively.

The properties of W_8-curvature tensor have been studied in [29, 27, 31, 15, 25, 28] and geometers obtained certain interesting results. Inspired by their work, in this paper, we have studied the properties of W_9-curvature tensor in Lorentzian para-Sasakian manifolds.

In 1982, R.S. Hamilton [14] introduced the concept of Ricci flow to identify a canonical metric on a smooth manifold. The Ricci flow describes the evolution of metrics on a Riemannian manifold M through the equation:

$$\frac{\partial}{\partial t}g = -2S,$$

where S represents the Ricci tensor. A special class of solutions to this equation, known as Ricci solitons, takes the form $g = \sigma(t)\,\psi_t^*g$ with the initial condition $g(0) = g$. Here, ψ_t denotes a family of diffeomorphisms on M, and

$\sigma(t)$ is a time-dependent scaling function. A Ricci soliton extends the concept of an Einstein metric. Following the definition in [5], a Ricci soliton on a manifold M is characterized by a triple $(g, \mathcal{Z}_2, \vartheta)$, where g is a Riemannian metric, $\mathcal{Z}_2$ is a vector field referred to as the potential vector field, and ϑ is a real scalar. These elements satisfy the equation:

$$\mathcal{L}_{\mathcal{Z}_2} g + 2S + 2\vartheta g = 0, \tag{1.3}$$

where $\mathcal{L}_{\mathcal{Z}_2}$ denotes the Lie derivative. Metrics that satisfy (1.3) are valuable in physics and are often called quasi-Einstein metrics [7, 6]. Compact Ricci solitons represent fixed points of the Ricci flow,

$$\frac{\partial}{\partial t} g = -2S,$$

when the space of metrics is projected onto its quotient by diffeomorphisms and scalings. These solitons frequently appear as blow-up limits of the Ricci flow on compact manifolds. Additionally, theoretical physicists have explored Ricci solitons in connection with string theory. Friedan [13] made the initial contribution in this area by examining some of its aspects.

Ricci solitons have been extensively explored by various researchers, including [11, 12, 14, 16] and many others. As a generalization of Ricci solitons, Cho and Kimura [8] introduced the concept of η-Ricci solitons, which has also been studied in [5] for Hopf hypersurfaces in complex space forms.

An η-Ricci soliton is defined by a quadruple $(g, \mathcal{Z}_2, \vartheta, \Psi)$, where $\mathcal{Z}_2$ is a vector field on M, ϑ and Ψ are real constants, and g is a Riemannian or pseudo-Riemannian metric that satisfies the equation:

$$\mathcal{L}_{\mathcal{Z}_2} g + 2S + 2\vartheta g + 2\Psi\, \eta \otimes \eta = 0. \tag{1.4}$$

Blaga [2, 3] and Prakasha et al. [26] have made notable contributions to the study of η-Ricci solitons. When $\Psi = 0$, the η-Ricci soliton $(g, \mathcal{Z}_2, \vartheta, \Psi)$ reduces to a standard Ricci soliton $(g, \mathcal{Z}_2, \vartheta)$. On the other hand, if $\Psi \neq 0$, the soliton is referred to as a proper η-Ricci soliton. For a detailed survey and additional references on the geometry of Ricci solitons on pseudo-Riemannian manifolds, we refer the reader to [1, 4, 23] and the literature cited therein.

The structure of this paper is outlined as follows: Section 2 deals with some preliminary concepts of Lorentzian para-Sasakian manifold. In Section 3, we discussed W_9-semisymmetric Lorentzian para-Sasakian manifold. In Section 4, we discussed ζ-W_9 flat Lorentzian para-Sasakian manifold. Furthurmore, in Section 5, we discussed W_9 flat Lorentzian para-Sasakian manifold. Moreover, in Section 6, we discussed W_9-recurrent Lorentzian para-Sasakian manifold. In Section 7, we studied ϕ-W_9 semisymmetric Lorentzian para-Sasakian manifold. Lastly, in Section 8, we explored η-Ricci solitons on Lorentzian para-Sasakian manifold satisfying $W_9(\zeta, \mathcal{F}_1) \cdot S = 0$.

2. Preliminaries

Let $\mathcal{M}^n$ be an n-dimensional differentiable manifold equipped with a $(1,1)$ tensor field ϕ, a contravariant vector field ζ, a covariant vector field η, and a Lorentzian metric g of type $(0,2)$. For each point p in $\mathcal{M}^n$, the metric g_p defines a non-degenerate inner product on the tangent space $T_pM \times T_pM$ mapping to $\mathbb{R}$ which satisfies the following properties

$$\phi^2(\mathcal{F}_1) = \mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta, \tag{2.1}$$

$$\eta(\zeta) = -1, \tag{2.2}$$

$$g(\mathcal{F}_1, \zeta) = \eta(\mathcal{F}_1), \tag{2.3}$$

$$g(\phi\mathcal{F}_1, \phi\mathcal{F}_2) = g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2)$$

for any vector fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{M}^n$. Such a structure (ϕ, ζ, η, g) is referred to as a Lorentzian almost paracontact structure, and the manifold $\mathcal{M}^n$ equipped with this structure is called a Lorentzian almost paracontact manifold [17].

$$\phi\zeta = 0, \quad \eta(\phi\mathcal{F}_1) = 0, \qquad \Omega(\mathcal{F}_1, \mathcal{F}_2) = \Omega(\mathcal{F}_2, \mathcal{F}_1)$$

where $\Omega(\mathcal{F}_1, \mathcal{F}_2) = g(\mathcal{F}_1, \phi\mathcal{F}_2)$.

Let $\{\, e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis such that $e_n = \zeta$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(\mathcal{F}_1, \mathcal{F}_2) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, \mathcal{F}_1)\mathcal{F}_2, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i)$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{n-1} = 1$, $\epsilon_n = -1$.

A Lorentzian almost paracontact manifold $\mathcal{M}^n$ admitting the structure (ϕ, ζ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(\mathcal{F}_1, \mathcal{F}_2) = \frac{1}{2}\left((\nabla_{\mathcal{F}_1}\eta)\mathcal{F}_2 + (\nabla_{\mathcal{F}_2}\eta)\mathcal{F}_1\right).$$

A Lorentzian almost paracontact manifold $\mathcal{M}^n$ admitting the structure (ϕ, ζ, η, g) is called Lorentzian para-Sasakian manifold if [17]

$$(\nabla_{\mathcal{F}_1}\phi)\mathcal{F}_2 = g(\phi\mathcal{F}_1, \phi\mathcal{F}_2)\zeta + \eta(\mathcal{F}_2)\phi^2\mathcal{F}_1.$$

In a Lorentzian para-Sasakian manifold the 1-form of η is closed. Also in [17], it is proved that if an n-dimensional Lorentzian manifold $(\mathcal{M}^n, g)$ admits a timelike unit vector field ζ such that the 1-form η associated to ζ is closed and satisfies

$$(\nabla_{\mathcal{F}_1}\nabla_{\mathcal{F}_2}\eta)\mathcal{F}_3 = g(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) + g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2) + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3),$$

then $\mathcal{M}^n$ admits a Lorentzian para-Sasakian structure. Further, on Lorentzian para-Sasakian manifold $\mathcal{M}^n(\phi, \zeta, \eta, g)$, the following relation holds [17]:

$$\eta(R(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3) = [g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2)], \tag{2.4}$$

$$S(\mathcal{F}_1, \zeta) = (n-1)\eta(\mathcal{F}_1), \tag{2.5}$$

$$\begin{aligned} S(\phi\mathcal{F}_1, \phi\mathcal{F}_2) &= S(\mathcal{F}_1, \mathcal{F}_2) + (n-1)\eta(\mathcal{F}_1)\eta(\mathcal{F}_2), \\ R(\mathcal{F}_1, \mathcal{F}_2)\zeta &= [\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2], \\ R(\zeta, \mathcal{F}_1)\mathcal{F}_2 &= g(\mathcal{F}_1, \mathcal{F}_2)\zeta - \eta(\mathcal{F}_2)\mathcal{F}_1, \\ (\nabla_{\mathcal{F}_1}\phi)(\mathcal{F}_2) &= [g(\mathcal{F}_1, \mathcal{F}_2)\zeta + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\zeta + \eta(\mathcal{F}_2)\mathcal{F}_1], \end{aligned}$$

for all vector fields $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}^n$. Here R denotes the curvature tensor on the manifold $\mathcal{M}^n$ and S denotes the Ricci tensor on $\mathcal{M}^n$. Although the vector fields η is closed in an Lorentzian para-Sasakian manifold, we have ([17, 18])

$$\begin{aligned} (\nabla_{\mathcal{F}_1}\eta)\mathcal{F}_2 &= \Omega(\mathcal{F}_1, \mathcal{F}_2), \\ \Omega(\mathcal{F}_1, \zeta) &= 0, \\ \nabla_{\mathcal{F}_1}\zeta &= \phi\mathcal{F}_1, \end{aligned} \tag{2.6}$$

for any vector fields $\mathcal{F}_1$, $\mathcal{F}_2$ on $\mathcal{M}^n$.

Definition 2.1. A Lorentzian para-Sasakian manifold $\mathcal{M}^n$ is said to be η-Einstein manifold if its Ricci tensor S is on the following form:

$$S(\mathcal{F}_1, \mathcal{F}_2) = \vartheta_1 g(\mathcal{F}_1, \mathcal{F}_2) + \vartheta_2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2),$$

for any vector fields $\mathcal{F}_1$, $\mathcal{F}_2$ on $\mathcal{M}^n$. Here ϑ_1, ϑ_2 are smooth functions on $\mathcal{M}^n$.

If ϑ_2=0, then $\mathcal{M}^n$ is an Einstein manifold.

Definition 2.2. Let W_9 be a $(1,3)$-type tensor. A semi-Riemannian manifold $(\mathcal{M}^n, g)$ is said to be W_9-recurrent if it satisfies

$$(\nabla_{\mathcal{Z}_1} W_9)(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = \alpha(\mathcal{Z}_1) W_9(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3,$$

for some non zero 1-form α.

Let $(M, \phi, \zeta, \eta, g)$ denote an almost paracontact metric manifold. Taking (1.4) in consideration and writing $\mathcal{L}_\zeta g$ in terms of Levi-Civita connection ∇, we obtain

$$\begin{aligned} 2S(\mathcal{F}_1, \mathcal{F}_2) = & - g(\nabla_{\mathcal{F}_1}\zeta, \mathcal{F}_2) - g(\mathcal{F}_1, \nabla_{\mathcal{F}_2}\zeta) \\ & - 2\vartheta g(\mathcal{F}_1, \mathcal{F}_2) - 2\Psi \eta(\mathcal{F}_1)\eta(\mathcal{F}_2), \end{aligned} \tag{2.7}$$

for any vector fields $\mathcal{F}_1$,$\mathcal{F}_2$ on $\mathcal{M}^n$.

Using (2.6) in (2.7), we have

$$S(\mathcal{F}_1, \mathcal{F}_2) = -g(\phi\mathcal{F}_1, \mathcal{F}_2) - \vartheta g(\mathcal{F}_1, \mathcal{F}_2) - \Psi\eta(\mathcal{F}_1)\eta(\mathcal{F}_2). \tag{2.8}$$

Setting $\mathcal{F}_2 = \zeta$ in (2.8), we obtain

$$S(\mathcal{F}_1, \zeta) = (\Psi - \vartheta)\eta(\mathcal{F}_1). \tag{2.9}$$

Putting $\mathcal{F}_1 = \zeta$ in (2.9), we have

$$S(\zeta, \zeta) = \vartheta - \Psi.$$

By virtue of (2.9), we have

$$Q\mathcal{F}_1 = (\Psi - \vartheta)\mathcal{F}_1.$$

By virtue of (2.5) in (2.9), we obtain

$$\Psi - \vartheta = n - 1. \tag{2.10}$$

3. Lorentzian para-Sasakian manifold admitting W_9-semisymmetric condition

In this section, we examine Lorentzian para-Sasakian manifold admitting W_9-semisymmetric condition.

DEFINITION 3.1. A Lorentzian para-Sasakian manifold is said to be W_9-semisymmetric if it satisfies

$$R(\mathcal{F}_1, \mathcal{F}_2)W_9(\mathcal{F}_3, \mathcal{Z}_1)\mathcal{Z}_2 = 0, \tag{3.1}$$

for all vector fields $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{Z}_1$ on $\mathcal{M}^n$.

Taking inner product of (3.1) with ζ, we obtain

$$R(\mathcal{F}_1, \mathcal{F}_2, W_9(\mathcal{F}_3, \mathcal{Z}_1)\mathcal{Z}_2, \zeta) = 0. \tag{3.2}$$

Using (2.4) in (3.2),we have

$$\eta(\mathcal{F}_1)W_9(\mathcal{F}_2, \mathcal{F}_3, \mathcal{Z}_1, \mathcal{Z}_2) - \eta(\mathcal{F}_2)W_9(\mathcal{F}_1, \mathcal{F}_3, \mathcal{Z}_1, \mathcal{Z}_2) = 0.$$

Since, $\eta(\mathcal{F}_1) \neq 0$ and $\eta(\mathcal{F}_2) \neq 0$, then it follows that

$$W_9(\mathcal{F}_2, \mathcal{F}_3, \mathcal{Z}_1, \mathcal{Z}_2) = 0 \qquad \text{and} \qquad W_9(\mathcal{F}_1, \mathcal{F}_3, \mathcal{Z}_1, \mathcal{Z}_2) = 0,$$

Hence from above discussion, we state the following theorem.

THEOREM 3.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold admitting W_9-semisymmetric condition, then the manifold is W_9-flat.*

4. ζ-W_9 FLAT LORENTZIAN PARA-SASAKIAN MANIFOLD

In this section, we examine Lorentzian para-Sasakian manifold admitting ζ-W_9 flatness condition.

DEFINITION 4.1. A Lorentzian para-Sasakian manifold is said to be ζ-W_9 flat if it satisfies

$$W_9(\mathcal{F}_1, \mathcal{F}_2)\zeta = 0, \tag{4.1}$$

for any vector fields $\mathcal{F}_1, \mathcal{F}_2$ on $\mathcal{M}^n$.

By virtue of (1.2) and using (4.1), we have

$$\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2 + \frac{1}{(n-1)}[S(\mathcal{F}_1, \mathcal{F}_2)\zeta - \eta(\mathcal{F}_2)Q\mathcal{F}_1] = 0. \tag{4.2}$$

Further, taking inner product of (4.2) with ζ and using (2.2), (2.3), we obtain

$$S(\mathcal{F}_1, \mathcal{F}_2) = -(n-1)\eta(\mathcal{F}_1)\eta(\mathcal{F}_2).$$

Hence from above discussion, we state the following theorem.

THEOREM 4.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold satisfying ζ-W_9 flat condition, then the manifold is a special type of η-Einstein manifold.*

5. Lorentzian para-Sasakian manifold admitting W_9-flat condition

In this section, we examine Lorentzian para-Sasakian manifold admitting W_9-flat condition.

DEFINITION 5.1. A Lorentzian para-Sasakian manifold is said to be W_9-flat if it satisfies

$$W_9(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = 0, \tag{5.1}$$

for any vector fields $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}^n$.

Taking inner product of (1.2) with ζ and using (5.1), we obtain

$$R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = \frac{1}{(n-1)}[g(\mathcal{F}_2, \mathcal{F}_3)S(\mathcal{F}_1, \zeta) - S(\mathcal{F}_1, \mathcal{F}_2)g(\mathcal{F}_3, \zeta)]. \tag{5.2}$$

By virtue of (2.3) and (2.5) in (5.2), we have

$$R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3). \tag{5.3}$$

But in Lorentzian para-Sasakian manifold, we have

$$R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2). \tag{5.4}$$

For (5.3) and (5.4) to hold simultaneously, we must have

$$R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = 0.$$

Hence from above discussion, we state the following theorem.

THEOREM 5.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold satisfying W_9-flat condition, then the manifold is flat.*

6. Lorentzian para-Sasakian manifold satisfying W_9-recurrent condition

In this section, we examine Lorentzian para-Sasakian manifold satisfying W_9-recurrent condition.

DEFINITION 6.1. A Lorentzian para-Sasakian manifold $(\mathcal{M}^n, g)$ is said to be W_9-recurrent if it satisfies

$$(\nabla_{\mathcal{Z}_1} W_9)(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = \alpha(\mathcal{Z}_1) W_9(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3,$$

for some non zero 1-form α.

Taking inner product of (1.2) with ζ, we have

$$\begin{aligned} W_9(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) &= R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) \\ &\quad + \frac{1}{(n-1)} \big[S(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) - g(\mathcal{F}_2, \mathcal{F}_3) S(\mathcal{F}_1, \zeta)\big]. \end{aligned} \tag{6.1}$$

Taking covariant derivative of (6.1) with respect to $\mathcal{Z}_1$ and using (1.2), we get

$$\begin{aligned} \nabla_{\mathcal{Z}_1} W_9(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) &= \nabla_{\mathcal{Z}_1} R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) \\ &\quad + \frac{1}{(n-1)} \big[\nabla_{\mathcal{Z}_1} S(\mathcal{F}_1, \mathcal{F}_2) g(\mathcal{F}_3, \zeta) - \nabla_{\mathcal{Z}_1} S(\mathcal{F}_1, \zeta) g(\mathcal{F}_2, \mathcal{F}_3)\big]. \end{aligned} \tag{6.2}$$

But, since it is known that

$$\nabla_{\mathcal{Z}_1} S(\mathcal{F}_1, \mathcal{F}_2) = \alpha(\mathcal{Z}_1) S(\mathcal{F}_1, \mathcal{F}_2), \tag{6.3}$$

and

$$\nabla_{\mathcal{Z}_1} W_9(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = \alpha(\mathcal{Z}_1) W_9(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta). \tag{6.4}$$

Therefore, using (6.3),(6.4) in (6.2), we obtain

$$\nabla_{\mathcal{Z}_1} R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta) = \alpha(\mathcal{Z}_1) R(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \zeta).$$

Hence from above discussion, we state the following theorem.

THEOREM 6.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold satifying W_9-recurrent condition.Then for the same recurrence parameter, the manifold is recurrent.*

7. Lorentzian para-Sasakian manifold satisfying ϕ-W_9-semisymmetric condition

In this section, we examine Lorentzian para-Sasakian manifold satisfying ϕ-W_9-semisymmetric condition.

DEFINITION 7.1. A Lorentzian para-Sasakian manifold $(\mathcal{M}^n, g)$ is said to satisfy ϕ-W_9-semisymmetric condition if

$$(W_9(\mathcal{F}_1, \mathcal{F}_2) \cdot \phi)\mathcal{F}_3 = 0, \tag{7.1}$$

for any vector fields $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}^n$.

From (7.1), we have

$$W_9(\mathcal{F}_1, \mathcal{F}_2)\phi\mathcal{F}_3 - \phi W_9(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = 0. \tag{7.2}$$

Replacing $\mathcal{F}_3$ by $\phi\mathcal{F}_3$ in (1.2) and using in (7.2), we obtain

$$\begin{aligned} g(\mathcal{F}_2, \phi\mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \phi\mathcal{F}_3)\mathcal{F}_2 + g(\mathcal{F}_1, \mathcal{F}_3)\phi\mathcal{F}_2 \\ - \frac{1}{(n-1)}[g(\mathcal{F}_2, \phi\mathcal{F}_3)Q\mathcal{F}_1] = 0. \end{aligned} \tag{7.3}$$

Taking inner product of (7.3) with $\mathcal{Z}_3$ and on simplification, we get

$$\begin{aligned} g(\mathcal{F}_2, \phi\mathcal{F}_3)g(\mathcal{F}_1, \mathcal{Z}_3) - g(\mathcal{F}_1, \phi\mathcal{F}_3)g(\mathcal{F}_2, \mathcal{Z}_3) \\ +g(\mathcal{F}_1, \mathcal{F}_3)g(\phi\mathcal{F}_2, \mathcal{Z}_3) - \frac{1}{(n-1)}g(\mathcal{F}_2, \phi\mathcal{F}_3)S(\mathcal{F}_1, \mathcal{Z}_3) = 0. \end{aligned} \tag{7.4}$$

Replacing $\mathcal{F}_3$ by $\phi\mathcal{F}_3$ in (7.4) and using (2.1), we obtain

$$\begin{aligned} \big[g(\mathcal{F}_2, \mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)\big]\Big[1 - \frac{1}{(n-1)}S(\mathcal{F}_1, \mathcal{Z}_3)\Big] \\ -\big[g(\mathcal{F}_1, \mathcal{F}_3) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_3)\big]g(\mathcal{F}_2, \mathcal{Z}_3) + g(\mathcal{F}_1, \phi\mathcal{F}_3)g(\phi\mathcal{F}_2, \mathcal{Z}_3) = 0. \end{aligned}$$

Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis such that $e_n = \zeta$. Substituting $\mathcal{F}_1 = \mathcal{Z}_3 = e_i$ and taking summation over i, where $1 \le i \le n$, we obtain

$$\Big[1 - \frac{r}{(n-1)}\Big]\big[g(\mathcal{F}_2, \mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)\big] = 0. \tag{7.5}$$

From (7.5), we arrive at following cases:

CASE I: If $\Big[1 - \frac{r}{(n-1)}\Big] = 0$, then we have $r = (n-1)$.

CASE II: If $g(\mathcal{F}_2, \mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3) = 0$, then replacing $\mathcal{F}_2$ by $Q\mathcal{F}_2$, we have

$$S(\mathcal{F}_2, \mathcal{F}_3) = -(n-1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).$$

Hence from above discussion, we state the following theorem.

THEOREM 7.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold satifying ϕ-W_9-semisymmetric condition. Then either the scalar curvature $r = (n-1)$ or $\mathcal{M}^n$ is a special type of η-Einstein manifold.*

8. η-Ricci solitons on Lorentzian para-Sasakian manifold satisfying $W_9(\zeta, \mathcal{F}_1) \cdot S = 0$

In this section, we examine η-Ricci solitons on Lorentzian para-Sasakian manifold satisfying $W_9(\zeta, \mathcal{F}_1) \cdot S = 0$. The condition that must be satisfied by S is given as [2]

$$S(W_9(\zeta, \mathcal{F}_1)\mathcal{F}_2, \mathcal{F}_3) + S(\mathcal{F}_2, W_9(\zeta, \mathcal{F}_1)\mathcal{F}_3) = 0, \tag{8.1}$$

for any vector fields $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}^n$.

By virtue of (1.2), we obtain

$$\begin{aligned} S(W_9(\zeta, \mathcal{F}_1)\mathcal{F}_2, \mathcal{F}_3) = (\Psi - \vartheta) \left[1 - \left(\frac{\Psi - \vartheta}{n-1}\right)\right] \eta(\mathcal{F}_3) g(\mathcal{F}_1, \mathcal{F}_2) \\ + \eta(\mathcal{F}_2) g(\phi\mathcal{F}_1, \mathcal{F}_3) + \vartheta\eta(\mathcal{F}_2) g(\mathcal{F}_1, \mathcal{F}_3) \\ + \Psi\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) + \left[\frac{\Psi - \vartheta}{n-1}\right] \eta(\mathcal{F}_1) S(\mathcal{F}_2, \mathcal{F}_3), \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} S(\mathcal{F}_2, W_9(\zeta, \mathcal{F}_1)\mathcal{F}_3) = (\Psi - \vartheta) \left[1 - \left(\frac{\Psi - \vartheta}{n-1}\right)\right] \eta(\mathcal{F}_2) g(\mathcal{F}_1, \mathcal{F}_3) \\ + \eta(\mathcal{F}_3) g(\phi\mathcal{F}_1, \mathcal{F}_2) + \vartheta\eta(\mathcal{F}_3) g(\mathcal{F}_1, \mathcal{F}_2) \\ + \Psi\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) + \left[\frac{\Psi - \vartheta}{n-1}\right] \eta(\mathcal{F}_1) S(\mathcal{F}_2, \mathcal{F}_3). \end{aligned} \tag{8.3}$$

Using (8.2) and (8.3) in (8.1), we have

$$\left[\frac{\Psi - \vartheta}{n-1}\right] [S(\mathcal{F}_2, \mathcal{F}_3) + (\Psi - \vartheta)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)] = 0. \tag{8.4}$$

By (8.4), we infer following cases:

Case I: If $\left[\frac{\Psi-\vartheta}{n-1}\right] = 0$, then we have $\Psi = \vartheta$.

Case II: If $\left[\frac{\Psi-\vartheta}{n-1}\right] \neq 0$, then we have

$$S(\mathcal{F}_2, \mathcal{F}_3) = -(\Psi - \vartheta)\eta(Y)\eta(\mathcal{F}_3).$$

Using (2.10), we have

$$S(\mathcal{F}_2, \mathcal{F}_3) = -(n-1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).$$

Hence from above discussion, we state the following theorem.

THEOREM 8.1. *Let $\mathcal{M}^n$ be an n-dimensional Lorentzian para-Sasakian manifold and $(g, \zeta, \vartheta, \Psi)$ be the η-Ricci soliton on $\mathcal{M}^n$, then either $\vartheta = \Psi$ or $\mathcal{M}^n$ is a special type of an η-Einstein manifold.*

9. Conclusion

This paper provides an in-depth study of various aspects of Lorentzian para-Sasakian manifolds, focusing on specific curvature conditions and symmetrical properties. In Section 2, we introduced preliminary concepts essential for understanding the structure of Lorentzian para-Sasakian manifolds. Section 3 explored the notion of W_9-semisymmetry, which is pivotal in characterizing these manifolds under certain curvature constraints. Section 4 discussed ζ-W_9 flat Lorentzian para-Sasakian manifolds, revealing conditions under which the curvature vanishes along the Reeb vector field ζ.

In Section 5, we examined W_9 flat manifolds, establishing the criteria for their curvature tensor to satisfy specific flatness conditions. Section 6 delved into W_9-recurrent structures, highlighting how these recurrence properties affect the geometric behavior of the manifold. Moving forward, Section 7 analyzed ϕ-W_9 semisymmetric Lorentzian para-Sasakian manifolds, focusing on the interplay between the structure tensor ϕ and curvature.

Finally, Section 8 investigated η-Ricci solitons on Lorentzian para-Sasakian manifolds, particularly under the condition $W_9(\zeta, \mathcal{F}_1) \cdot S = 0$. This section underscores the relevance of solitons in the evolution of geometric flows on these manifolds.

Throughout the paper, we explored various curvature conditions and their impact on the geometric properties of Lorentzian para-Sasakian manifolds. These findings contribute to a deeper understanding of such manifolds, potentially leading to further research in geometric structures and their applications in mathematical physics.

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