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Preservation Results for New Spectral Properties

Hassan Zariouh

Département de Mathématiques, Centre Régional pour les Métiers de l'Éducation et de la Formation de la région de l'Oriental (CRMEFO), PO Box 458, Bodir, 60000 Oujda, Morocco

h.zariouh@yahoo.fr

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Abstract: A bounded linear operator T is said to satisfy property $(SBaw)$ if $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T) = E_a^0(T)$, where $\sigma_a(T)$ is the approximate point spectrum of T, $\sigma_{SBF_+^-}(T)$ is the upper semi-B-Weyl spectrum of T and $E_a^0(T)$ is the set of all eigenvalues of T of finite multiplicity that are isolated in its approximate point spectrum. In this paper we give a characterization of this spectral property for a bounded linear operator having SVEP on the complementary of its upper semi-B-Weyl spectrum, and we study its stability under commuting Riesz-type perturbations. Analogous results are obtained for the properties (SBb), $(SBab)$ and (SBw). The theory is exemplified in the case of some special classes of operators.

Key words: a-Browder's theorem, upper semi-Weyl spectrum, SVEP, Riesz operator.

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1. Introduction and preliminaries

For T in the Banach algebra $L(X)$ of bounded linear operators acting on a Banach space X, we will denote by $\sigma(T)$ the spectrum of T, by $\sigma_a(T)$ the approximate point spectrum of T, by $\mathcal{N}(T)$ the null space of T, by $n(T)$ the nullity of T, by $\mathcal{R}(T)$ the range of T and by $d(T)$ its defect. If $\mathcal{R}(T)$ is closed and $n(T) < \infty$ (resp., $d(T) < \infty$) then T is called an *upper semi-Fredholm* (resp., a *lower semi-Fredholm*) operator and its index is defined by $\text{ind}(T) = n(T)-d(T)$. An *upper semi-Weyl* operator is an upper semi-Fredholm operator of index less or equal than zero. The upper semi-Weyl spectrum is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Weyl operator}\}$.

For a bounded linear operator T and $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ be the restriction of T to $\mathcal{R}(T^n)$. $T \in L(X)$ is said to be *upper semi-B-Weyl* if for some integer $n \geq 0$ the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is upper semi-Weyl; its index is defined as the index of the upper semi-Weyl operator $T_{[n]}$. The respective *upper semi-B-Weyl spectrum* is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not an upper semi-B-Weyl operator$\}$.

The *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to [11], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has finite ascent and finite descent, and in this case they are equal. We recall that a complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $a(T - \lambda I) < \infty$ and $R(T^{a(T-\lambda I)+1})$ is closed. For further definitions, we refer the reader to [1] and [6]. In addition, we summarize in the following list the usual notations and symbols needed later.

Notations and symbols

$\mathcal{F}(X)$: the ideal of finite rank operators in $L(X)$,
$\mathcal{K}(X)$: the ideal of compact operators in $L(X)$,
$\mathcal{N}(X)$: the class of nilpotent operators on X,
$\mathcal{Q}(X)$: the class of quasi-nilpotent operators on X,
$\mathcal{R}(X)$: the class of Riesz operators acting on X,
$\operatorname{iso} A$: isolated points of a subset $A \subset \mathbb{C}$,
$\operatorname{acc} A$: accumulations points of a subset $A \subset \mathbb{C}$,
$D(0,1)$: the closed unit disc in $\mathbb{C}$,
$C(0,1)$: the unit circle of $\mathbb{C}$,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_p(T)$: eigenvalues of T,
$\sigma_p^0(T)$: eigenvalues of T of finite multiplicity,
$E^0(T) := \operatorname{iso} \sigma(T) \cap \sigma_p^0(T)$,
$E(T) := \operatorname{iso} \sigma(T) \cap \sigma_p(T)$,
$E_a^0(T) := \operatorname{iso} \sigma_a(T) \cap \sigma_p^0(T)$,
$E_a(T) := \operatorname{iso} \sigma_a(T) \cap \sigma_p(T)$,
$\sigma_b(T) = \sigma(T) \setminus \Pi^0(T)$: Browder spectrum of T,
$\sigma_{ub}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$: upper-Browder spectrum of T,
$\sigma_{SF_+^-}(T)$: upper semi-Weyl spectrum of T,
$\sigma_{SBF_+^-}(T)$: upper semi-B-Weyl spectrum of T,
the symbol $\bigsqcup$ stands for the disjoint union.
It is easily to verify that $E(T) \subset E_a(T)$, $\Pi(T) \subset \Pi_a(T) \subset E_a(T)$.

Definition 1.1. [4, 14, 15] Let $T \in L(X)$. T is said to satisfy

i) a-Weyl's theorem if $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup E_a^0(T)$;
ii) a-Browder's theorem if $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup \Pi_a^0(T)$;
iii) property (b) if $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup \Pi^0(T)$;
iv) property (w) if $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup E^0(T)$.

Definition 1.2. [3] Let $T \in L(X)$. We say that:

i) T satisfies property (SBw) if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \bigsqcup E^0(T)$;
ii) T satisfies property (SBb) if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \bigsqcup \Pi^0(T)$;
iii) T satisfies property $(SBaw)$ if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \bigsqcup E_a^0(T)$;
iv) T satisfies property $(SBab)$ if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \bigsqcup \Pi_a^0(T)$.

The relationship between properties and theorems given in the precedent definitions was studied in [3], and is summarized in the following diagram. (arrows signify implications and numbers near the arrows are references to the bibliography therein).

$$\begin{array}{ccccccc}
\text{a-Weyl's theorem} & \xleftarrow[{[3]}]{} & (SBaw) & & (SBw) & \xrightarrow{[3]} & (w) \\
\big\downarrow{\scriptstyle [10]} & & \big\downarrow{\scriptstyle [3]} & & \big\downarrow{\scriptstyle [3]} & & \big\downarrow{\scriptstyle [4]} \\
\text{a-Browder's theorem} & \xleftarrow[{[3]}]{} & (SBab) & \xleftarrow[{[3]}]{} & (SBb) & \xrightarrow[{[3]}]{} & (b)
\end{array}$$

We recall that the two properties $(SBaw)$ and (SBw) are independent, see [3, p. 276]. Moreover, in [3] counterexamples were given to show that the reverse of each implication in the diagram is not true. Nonetheless, it was proved that under some additional hypothesis, these implications are equivalences as we can see in the next theorem.

Theorem 1.3. [3] *Let* $T \in L(X)$.

i) *Property* $(SBaw)$ *holds for* T *if and only if a-Weyl's theorem holds for* T *and* $\sigma_{SF_+^-}(T) \setminus \sigma_{SBF_+^-}(T) = \emptyset$.
ii) *Property* $(SBab)$ *holds for* T *if and only if a-Browder's theorem holds for* T *and* $\sigma_{SF_+^-}(T) \setminus \sigma_{SBF_+^-}(T) = \emptyset$.

iii) *Property* (SBb) *holds for* T *if and only if property* (b) *holds for* T *and* $\sigma_{SF_+^-}(T)\setminus\sigma_{SBF_+^-}(T)=\emptyset$.

iv) *Property* (SBw) *holds for* T *if and only if property* (w) *holds for* T *and* $\sigma_{SF_+^-}(T)\setminus\sigma_{SBF_+^-}(T)=\emptyset$.

v) *Property* (SBw) *holds for* T *if and only if property* (SBb) *holds for* T *and* $E^0(T)=\Pi^0(T)$.

vi) *Property* (SBb) *holds for* T *if and only if property* $(SBab)$ *holds for* T *and* $\Pi^0(T)=\Pi_a(T)$.

vii) *Property* $(SBaw)$ *holds for* T *if and only if property* $(SBab)$ *holds for* T *and* $E_a^0(T)=\Pi_a(T)$.

For every $T\in L(X)$ we know that $\sigma_{SBF_+^-}(T)\subset\sigma_{SF_+^-}(T)$, but generally this inclusion is proper. Indeed, let T on $\ell^2(\mathbb{N})$ defined by $T(x_1,x_2,\ldots)=(0,\frac{x_1}{2},0,0,\ldots)$, then $\sigma_{SBF_+^-}(T)=\emptyset\subsetneq\sigma_{SF_+^-}(T)=\{0\}$. In the following lemma, we explicit the defect set $\sigma_{SF_+^-}(T)\setminus\sigma_{SBF_+^-}(T)$.

Lemma 1.4. *(See also [8]) Let* $T\in L(X)$. *Then* $\sigma_{SF_+^-}(T)=\sigma_{SBF_+^-}(T)\cup \operatorname{iso}\sigma_{SF_+^-}(T)$.

Proof. Let $\lambda_0\in\sigma_{SF_+^-}(T)\setminus\sigma_{SBF_+^-}(T)$ be arbitrary, then $T-\lambda_0I$ is an upper semi-B-Weyl operator. From the punctured neighborhood theorem for upper semi-B-Weyl operators, there exists $\varepsilon>0$ such that if $0<|\mu|<\varepsilon$, then $T-\lambda_0I-\mu I$ is an upper semi-Weyl operator and $\operatorname{ind}(T-\lambda_0I-\mu I)=\operatorname{ind}(T-\lambda_0I)$. Thus for every scalar z such that $0<|z-\lambda_0|<\varepsilon$, we have $T-\lambda_0I-(z-\lambda_0)I=T-zI$ is an upper semi-Weyl operator with $\operatorname{ind}(T-zI)\leq 0$. This implies that $D(\lambda_0,\varepsilon)\cap\sigma_{SF_+^-}(T)=\{\lambda_0\}$ and as $\lambda_0\in\sigma_{SF_+^-}(T)$, then $\lambda_0\in\operatorname{iso}\sigma_{SF_+^-}(T)$. Hence $\sigma_{SF_+^-}(T)=\sigma_{SBF_+^-}(T)\cup\operatorname{iso}\sigma_{SF_+^-}(T)$. ∎

Corollary 1.5. *Let* $T\in L(X)$ *such that* $\operatorname{iso}\sigma_{SF_+^-}(T)=\emptyset$. *The following statements hold.*

i) T *satisfies property* $(SBaw)$ *if and only if* T *satisfies a-Weyl's Theorem.*

ii) T *satisfies property* $(SBab)$ *if and only if* T *satisfies a-Browder's Theorem.*

iii) T *satisfies property* (SBb) *if and only if* T *satisfies property* (b).

iv) T *satisfies property* (SBw) *if and only if* T *satisfies property* (w).

Proof. It's a consequence of Theorem 1.3 and Lemma 1.4. ∎

The paper is organized as follows: after giving an introduction and some preliminaries in the first section, we characterize in the second section the properties (SBw), $(SBaw)$, $(SBab)$ and (SBb) for bounded linear operators having SVEP on the complementary of the upper semi-B-Weyl spectrum. In the third section, we study the preservation of properties (SBw) and $(SBaw)$ under Riesz-type perturbations. Similar results are obtained for (SBb) and $(SBab)$ in the fourth section. Several examples are given in each section to show that the results obtained fail without adequate hypothesis.

2. New spectral properties and SVEP

The following property has relevant role in local spectral theory: a bounded linear operator $T \in L(X)$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the function $f \equiv 0$ is the only analytic solution of the equation $(T-\mu I)f(\mu) = 0$, $\forall \mu \in U_\lambda$. We denote by $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T$ does not have SVEP at $\lambda\}$ and we say that T has SVEP if $\mathcal{S}(T) = \emptyset$. We say that T has SVEP on $A \subset \mathbb{C}$, if T has SVEP at every $\lambda \in A$. (For more details about this property, we refer the reader to [12]).

Theorem 2.1. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$ then T satisfies property $(SBab)$ if and only if $\Pi_a(T) = \Pi_a^0(T)$; where $\sigma_{SBF_+^-}(T)^C$ is the complement of the upper semi-B-Weyl spectrum of T.*

Proof. $\Rightarrow$) Assume that T satisfies property $(SBab)$. Then $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset \Pi_a^0(T)$. As the opposite inclusion is always true, it follows that $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$. But this is equivalent from [2, Theorem 2.2] to say that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$. Hence $\Pi_a(T) = \Pi_a^0(T)$. Observe that in this implication, the condition of SVEP for T or T^* is not necessary.

$\Leftarrow$) Assume that $\Pi_a(T) = \Pi_a^0(T)$. If T has SVEP on $\sigma_{SBF_+^-}(T)^C$, then from [1, Theorem 2.4], T satisfies generalized a-Browder's theorem $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$. Therefore T satisfies property $(SBab)$. If T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$, then from [5, Corollary 2.7], T satisfies generalized a-Browder's theorem and hence it satisfies property $(SBab)$. ∎

Remark 2.2. The assumption T or T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$ is essential as shown in the next example.

Define the operator U on $\ell^2(\mathbb{N})$ by $U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. On $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, put $T = U \oplus U^*$. Since $\sigma_a(U) = \sigma_{SBF_+^-}(U) = C(0,1)$ and $\sigma_a(U^*) = \sigma_{SBF_+^-}(U^*) = D(0,1)$, it follows that $\sigma_a(T) = D(0,1)$ and hence $\Pi_a(T) = \Pi_a^0(T) = \emptyset$. But as $n(T) = d(T) = 1$, $0 \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Thus property $(SBab)$ does not hold for T. Notice that T and T^* do not have SVEP at 0 which lies in $\sigma_{SBF_+^-}(T)^C$, since $\mathcal{S}(T) = \mathcal{S}(T^*) = \mathcal{S}(U^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$.

Corollary 2.3. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$, then T satisfies property (SBb) if and only if $\Pi^0(T) = \Pi_a(T)$.*

Proof. It's a consequence of the precedent theorem and [3, Corollary 2.11]. (Note that the direct implication is always true (see [3, Corollary 2.11]). ∎

Corollary 2.4. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$, then*

i) *T satisfies property $(SBaw)$ if and only if $\Pi_a(T) = E_a^0(T)$.*
ii) *T satisfies property (SBw) if and only if $\Pi_a(T) = E^0(T)$.*

Proof. i) If T satisfies $(SBaw)$ then from Theorem 1.3, $\Pi_a(T) = E_a^0(T)$. Conversely, if $\Pi_a(T) = E_a^0(T)$, then $\Pi_a(T) = E_a^0(T) = \Pi_a^0(T)$. From Theorem 2.1 it follows that T satisfies property $(SBab)$ and hence it satisfies property $(SBaw)$.

ii) If T satisfies (SBw) then from Theorem 1.3, $E^0(T) = \Pi_a(T)$. Conversely, if $E^0(T) = \Pi_a(T)$, then $E^0(T) = \Pi_a(T) = \Pi_a^0(T)$. From Theorem 2.1 we conclude that T satisfies property (SBw). ∎

Remark 2.5. The assumption T or T^* has SVEP on $\sigma_{SBF_+^-}(T)^C$, is essential in corollaries 2.3 and 2.4. Indeed, the operator T given in Remark 2.2 does not satisfy property $(SBab)$ and hence it does not satisfy the properties (SBb), $(SBaw)$ and (SBw); though we have $\Pi_a(T) = E^0(T) = E_a^0(T) = \Pi^0(T)$.

3. Properties $(SBaw)$, (SBw) and Riesz-type perturbations

We recall that an operator $R \in L(X)$ is said to be *Riesz* if $R - \mu I$ is Fredholm for every non-zero complex μ, that is, $\pi(R)$ is quasinilpotent in the Calkin algebra $C(X) = L(X)/\mathcal{K}(X)$ where π is the canonical mapping of $L(X)$ into $C(X)$.

We denote by $\mathcal{F}^0(X)$, the class of finite rank power operators as follows:

$$\mathcal{F}^0(X) = \{S \in L(X) : S^n \in \mathcal{F}(X) \text{ for some } n \in \mathbb{N}\}.$$

Clearly,

$$\mathcal{F}(X) \cup \mathcal{N}(X) \subset \mathcal{F}^0(X) \subset \mathcal{R}(X), \text{ and } \mathcal{K}(X) \cup \mathcal{Q}(X) \subset \mathcal{R}(X).$$

We start this section by the following nilpotent perturbation result.

Proposition 3.1. *Let $T \in L(X)$ and let $N \in \mathcal{N}(X)$ which commutes with T. Then T satisfies property (s) if and only if $T + N$ satisfies property (s), where $(s) \in \{(SBw), (SBb), (SBab), (SBaw)\}$.*

Proof. Since N is nilpotent and commutes with T, we know that $\sigma(T + N) = \sigma(T)$ and $\sigma_a(T + N) = \sigma_a(T)$. From the proof of [6, Theorem 3.5], it follows that $0 < n(T + N) \Leftrightarrow 0 < n(T)$ and $n(T + N) < \infty \Leftrightarrow n(T) < \infty$. Thus $E^0_a(T + N) = E^0_a(T)$, $E(T + N) = E(T)$, $E_a(T + N) = E_a(T)$ and $E^0(T+N) = E^0(T)$. We also have from [7, Lemma 2.2] that $\Pi(T+N) = \Pi(T)$ which implies that $\Pi^0(T + N) = \Pi^0(T)$. From [18, Corollary 3.8] we know that $\Pi_a(T + N) = \Pi_a(T)$ and so $\Pi^0_a(T + N) = \Pi^0_a(T)$. On the other hand, $\sigma_{SBF^-_+}(T + N) = \sigma_{SF^-_+}(T)$, see [18, Corollary 3.1]. This finishes the proof. ∎

Remark 3.2. We notice that the assumption of commutativity in Proposition 3.1 is crucial.

1) Let T and N be defined on $\ell^2(\mathbb{N})$ by

$$T(x_1, x_2, \ldots) = \Big(0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots\Big) \text{ and } N(x_1, x_2, \ldots) = \Big(0, \frac{-x_1}{2}, 0, 0, \ldots\Big).$$

Clearly N is nilpotent and does not commute with T. The properties $(SBaw)$ and (SBw) are satisfied by T, since $\sigma_a(T) = \{0\} = \sigma_{SBF^-_+}(T)$ and $E^0_a(T) = \emptyset$. But $T + N$ does not satisfy neither property (SBw) nor property $(SBaw)$ as we have $\sigma_a(T+N) = \sigma_{SBF^-_+}(T+N) = \{0\}$ and $\{0\} = E^0(T+N) = E^0_a(T+N)$.

2) Let T and N be defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots) \text{ and } N(x_1, x_2, \ldots) = (0, -x_1, 0, 0, \ldots).$$

N is nilpotent and $TN \neq NT$. Moreover, $\sigma_a(T) = C(0,1)$, $\sigma_{SBF_+^-}(T) = C(0,1)$, $\sigma(T) = \sigma(T+N) = D(0,1)$, $\Pi^0(T) = \emptyset$, $\sigma_a(T+N) = C(0,1) \cup \{0\}$, $\sigma_{SBF_+^-}(T+N) = C(0,1)$, and $\Pi^0(T+N) = \emptyset$. So T satisfies property (SBb), but $T+N$ does not satisfy property (SBb).

Corollary 3.3. *Let $T \in \mathcal{Q}(X)$ be an injective quasi-nilpotent and let $F \in \mathcal{F}(X)$ which commutes with T. Then T satisfies property (s) if and only if $T+F$ satisfies property (s), where $(s) \in \{(SBw), (SBb), (SBab), (SBaw)\}$.*

Proof. If T is injective, as TF is a finite rank quasi-nilpotent operator, then TF is a nilpotent operator. Since T is injective, then F is nilpotent. Thus the result follows from Proposition 3.1. ∎

The stability of properties $(SBaw)$ and (SBw) showed in Proposition 3.1 cannot be extended to commuting quasi-nilpotent operators, as we can see in the next example.

Example 3.4. We consider the operators T and R defined on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by

$$T = 0 \oplus Q \text{ and } R = Q \oplus 0,$$

where Q is defined on $\ell^2(\mathbb{N})$ by $Q(x_1, x_2, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Clearly R is compact and quasi-nilpotent and verifies $TR = RT = 0$. On the other hand, T satisfies properties (SBw) and $(SBaw)$, because $\sigma_a(T) = \{0\} = \sigma_{SBF_+^-}(T)$ and $E_a^0(T) = \emptyset$. But $T+R = Q \oplus Q$ does not satisfy neither property (SBw) nor property $(SBaw)$, since $\sigma_a(T+R) = \{0\} = \sigma_{SBF_+^-}(T+R)$ and $E^0(T+R) = E_a^0(T+R) = \{0\}$. Note that here $\Pi_a(T+R) = \emptyset$.

However, in Theorem 3.6 below we give necessary and sufficient conditions to ensure the stability of these properties under commuting perturbations by Riesz operators which are not necessary nilpotent. The case of nilpotent operators is studied in Proposition 3.1. But before that we need the following lemma in the proof of the next main results.

Lemma 3.5. *Let $T \in L(X)$. If $S \in \mathcal{F}^0(X)$ and $R \in \mathcal{R}(X)$ are commuting operators with T, then the following statements hold.*

i) *T satisfies a-Browder's theorem if and only if $T+R$ satisfies a-Browder's theorem.*
ii) *If T satisfies property $(SBab)$, then $\Pi_a(T+S) = \Pi_a^0(T+S)$. In particular, this equality holds if T satisfies property $(SBaw)$ or (SBb).*

Proof. i) As T satisfies a-Browder's theorem, then $\sigma_{ub}(T) = \sigma_{SF_+^-}(T)$. Since $TR = RT$ then from [16, Theorem 7] we have $\sigma_{ub}(T+R) = \sigma_{ub}(T)$ and from [17, Proposition 5] we have $\sigma_{SF_+^-}(T) = \sigma_{SF_+^-}(T+R)$. So $\sigma_{ub}(T+R) = \sigma_{SF_+^-}(T+R)$. Thus $T+R$ satisfies a-Browder's theorem, and hence $T+R$ satisfies generalized a-Browder's theorem. Conversely, assume that $T+R$ satisfies a-Browder's theorem. Since $(T+R)R = R(T+R)$ and $T = (T+R) - R$, we conclude similarly.

ii) Since the inclusion $\Pi_a^0(T+S) \subset \Pi_a(T+S)$ is always true. To prove opposite inclusion, let $\lambda \in \Pi_a(T+S)$. As T satisfies property $(SBab)$, then from [3, Theorem 2.14] we have $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. Since $S \in \mathcal{F}^0(X)$ and $TS = ST$, then from [18, Theorem 2.8] we have $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+S)$. Hence $\lambda \notin \sigma_{SBF_+^-}(T+S) = \sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T+S)$. So $n(T+S-\lambda I) < \infty$. In particular, if T satisfies property $(SBaw)$ or (SBb) then it satisfies property $(SBab)$. ∎

THEOREM 3.6. *Let $R \in \mathcal{R}(X)$ and let $T \in L(X)$ which commutes with R. We have:*

i) *If T satisfies property (SBw), then $T+R$ satisfies property (SBw) if and only if $\Pi_a(T+R) = E^0(T+R)$.*
ii) *If T satisfies property $(SBaw)$, then $T+R$ satisfies property $(SBaw)$ if and only if $\Pi_a(T+R) = E_a^0(T+R)$.*

Proof. i) If $T+R$ satisfies (SBw), then from Theorem 1.3 we have $E^0(T+R) = \Pi_a(T+R)$. Conversely, suppose that $E^0(T+R) = \Pi_a(T+R)$. Since T satisfies (SBw) then from [3], it satisfies a-Browder's theorem. From Lemma 3.5, $T+R$ satisfies generalized a-Browder's theorem, that is $\sigma_a(T+R) \setminus \sigma_{SBF_+^-}(T+R) = \Pi_a(T+R)$. So $T+R$ satisfies property (SBw).

ii) If $T+R$ satisfies $(SBaw)$, then from Theorem 1.3 we have $E_a^0(T+R) = \Pi_a(T+R)$. Conversely, suppose that $E_a^0(T+R) = \Pi_a(T+R)$. Since T satisfies $(SBaw)$ then from Theorem 1.3, it satisfies a-Browder's theorem. Hence $T+R$ satisfies generalized a-Browder's theorem $\sigma_a(T+R) \setminus \sigma_{SBF_+^-}(T+R) = \Pi_a(T+R)$. So $\sigma_a(T+R) \setminus \sigma_{SBF_+^-}(T+R) = E_a^0(T+R)$. ∎

Now, if we restrict to the class $\mathcal{F}^0(X)$ we obtain the following perturbation result concerning property $(SBaw)$.

THEOREM 3.7. *Let* $S \in \mathcal{F}^0(X)$. *If* $T \in L(X)$ *satisfies property* $(SBaw)$ *and commutes with* S, *then the following statements are equivalent.*

i) $T+S$ *satisfies property* $(SBaw)$,
ii) $\Pi_a(T+S) = E_a^0(T+S)$,
iii) $E_a^0(T+S) \cap \sigma_a(T) \subset \Pi_a^0(T)$.

Proof. i) $\Leftrightarrow$ ii) Since $\mathcal{F}^0(X) \subset \mathcal{R}(X)$, this follows from Theorem 3.6.

ii) $\Rightarrow$ iii) Suppose that $\Pi_a(T+S) = E_a^0(T+S)$ and let $\lambda_0 \in E_a^0(T+S) \cap \sigma_a(T)$ be arbitrary. Then $\lambda_0 \in \Pi_a^0(T+S) \cap \sigma_a(T)$ and so $\lambda_0 \notin \sigma_{ub}(T+S) = \sigma_{ub}(T)$. Thus $\lambda_0 \in \Pi_a^0(T)$. This proves that $E^0(T+S) \cap \sigma_a(T) \subset \Pi_a^0(T)$.

iii) $\Rightarrow$ ii) Suppose that $E_a^0(T+S) \cap \sigma_a(T) \subset \Pi_a^0(T)$. Firstly, we show that $E_a^0(T+S) \subset \Pi_a(T+S)$. Let $\mu_0 \in E_a^0(T+S)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma_a(T)$. Then $\mu_0 \in E_a^0(T+S) \cap \sigma_a(T) \subset \Pi_a^0(T)$. It follows that $\mu_0 \notin \sigma_{ub}(T) = \sigma_{ub}(T+S)$ and since $\mu_0 \in \sigma_a(T+S)$, then $\mu_0 \in \Pi_a(T+S)$. The second case is $\mu_0 \notin \sigma_a(T)$. This implies that $\mu_0 \notin \sigma_{ub}(T) = \sigma_{ub}(T+S)$. Thus $\mu_0 \in \Pi_a^0(T+S) \subset \Pi_a(T+S)$. Consequently, $E_a^0(T+S) \subset \Pi_a(T+S)$. From Lemma 3.5, we conclude that $\Pi_a(T+S) = E_a^0(T+S)$. ∎

The following example proves that, in general, property (SBw) is not preserved under commuting finite rank power perturbations.

EXAMPLE 3.8. On $\ell^2(\mathbb{N})$, let U defined in Remark 2.2. For fixed $0 < \varepsilon < 1$, let F_ε ba a finite rank operator defined on $\ell^2(\mathbb{N})$ by $F_\epsilon(x_1, x_2, x_3, \dots) = (-\varepsilon x_1, 0, 0, 0, \dots)$. We consider the operators T and F defined by $T = U \oplus I$ and $F = 0 \oplus F_\varepsilon$, respectively. Then F is a finite rank operator and $TF = FT$. Moreover,

$$\sigma(T) = \sigma(U) \cup \sigma(I) = D(0,1), \quad \sigma_a(T) = \sigma_a(U) \cup \sigma_a(I) = C(0,1),$$

$$\sigma_{SBF_+^-}(T) = C(0,1), \quad \sigma(T+F) = \sigma(U) \cup \sigma(I+F_\varepsilon) = D(0,1),$$

$$\sigma_{SBF_+^-}(T+F) = C(0,1) \quad \text{and}$$

$$\sigma_a(T+F) = \sigma_a(U) \cup \sigma_a(I+F_\varepsilon) = C(0,1) \cup \{1-\varepsilon\}.$$

Moreover, we have $E^0(T) = \emptyset$ and $E^0(T+F) = \emptyset$. Thus T satisfies property (SBw), but $T+F$ does not satisfy property (SBw).

An operator $T \in L(X)$ is said to be finitely polaroid if $\operatorname{iso}\sigma(T) = \Pi^0(T)$, and is said to be finitely a-polaroid if $\operatorname{iso}\sigma_a(T) = \Pi^0_a(T)$.

LEMMA 3.9. *Let $T \in L(X)$ and let $F \in \mathcal{F}(X)$ which commutes with T. Then*

i) *T is finitely polaroid if and only if $T + F$ is finitely polaroid.*

ii) *T is finitely a-polaroid if and only if $T + F$ is finitely a-polaroid.*

Proof. i) Let T be finitely polaroid and $F \in \mathcal{F}(X)$. Then $\operatorname{acc}\sigma(T) = \sigma_b(T)$. Since F commutes with T, from [16, Corollary 8] we have $\sigma_b(T + F) = \sigma_b(T)$ and from [13, Lemma 2.1] we know that $\operatorname{acc}\sigma(T + F) = \operatorname{acc}\sigma(T)$. So $\sigma_b(T + F) = \operatorname{acc}\sigma(T + F)$ and $T + F$ is finitely polaroid. The proof of the reverse implication is similar, since $T = (T + F) - F$ and $T + F$ commutes with $-F$.

ii) Proof similar to the first assertion since $\operatorname{acc}\sigma_a(T + F) = \operatorname{acc}\sigma_a(T)$, see [9, Theorem 3.2] and $\sigma_{ub}(T + F) = \sigma_{ub}(T)$. ∎

COROLLARY 3.10. *Let $T \in L(X)$ and let $F \in \mathcal{F}(X)$ commutes with T. If T is finitely a-polaroid, then T satisfies property $(SBaw)$ if and only if $T + F$ satisfies property $(SBaw)$.*

Proof. Suppose that T satisfies property $(SBaw)$. Let $\lambda_0 \in E^0_a(T + F) \cap \sigma_a(T)$ be arbitrary, then $\lambda_0 \notin \operatorname{acc}\sigma_a(T + F) = \operatorname{acc}\sigma_a(T)$. So $\lambda_0 \in \operatorname{iso}\sigma_a(T) = \Pi^0_a(T)$. Hence $E^0_a(T + F) \cap \sigma_a(T) \subset \Pi^0_a(T)$, but this is equivalent by Theorem 3.7 to say that $T + F$ satisfies property $(SBaw)$. The proof of the reverse is similar, since $T + F$ is finitely a-polaroid. ∎

4. PROPERTIES $(SBab)$, (SBb) AND RIESZ-TYPE PERTURBATIONS

We begin this section with the following proposition in which, we improve Proposition 3.1 and show that the property $(SBab)$ is stable under commuting perturbations by operators of finite rank power.

PROPOSITION 4.1. *If $T \in L(X)$ satisfies property $(SBab)$ and if $S \in \mathcal{F}^0(X)$ commutes with T, then $T + S$ satisfies property $(SBab)$. In particular, if $S \in \mathcal{F}(X)$ and commutes with T then $T + S$ satisfies property $(SBab)$.*

Proof. Since $S \in \mathcal{F}^0(X)$ and $ST = TS$, then from Lemma 3.5 we have $\Pi_a(T + S) = \Pi^0_a(T + S)$. As T satisfies property $(SBab)$, then from Theorem 1.3, it satisfies generalized a-Browder's theorem. Lemma 3.5 implies that

$T+S$ satisfies generalized a-Browder's theorem. Thus $\sigma_a(T+S)\setminus\sigma_{SBF_+^-}(T+S)=\Pi_a(T+S)=\Pi_a^0(T+S)$. So $T+S$ satisfies property $(SBab)$. ∎

Remark 4.2. We cannot expect that the result announced in Proposition 4.1 remains correct in the case of property (SBb). For this, if we consider the operators T and F defined in Example 3.8, then T satisfies property (SBb), since $\sigma_a(T)=\sigma_{SBF_+^-}(T)\bigsqcup\Pi^0(T)=C(0,1)$. But $T+F$ does not satisfy property (SBb), since $\sigma_{SBF_+^-}(T+F)\bigsqcup\Pi^0(T+F)=C(0,1)$ and $\sigma_a(T+F)=C(0,1)\cup\{1-\varepsilon\}$.

As we have observed in the precedent section, we also cannot extend Proposition 3.1 concerning properties $(SBab)$ and (SBb) to commuting quasi-nilpotent perturbations, as shown in the next example.

Example 4.3. Let T be the operator defined on $\ell^2(\mathbb{N})$ by $T(x_1,x_2,\ldots)=(\frac{x_2}{2},\frac{x_3}{3},\ldots)$. Put $R=-T$, clearly R is quasi-nilpotent, compact and commutes with T. Moreover, we have $\sigma_a(T)=\{0\}=\sigma_{SBF_+^-}(T)$ and $\Pi^0(T)=\Pi_a^0(T)=\emptyset$. It follows that T satisfies properties $(SBab)$ and (SBb). But $T+R=0$ does not satisfy neither property $(SBab)$ nor property (SBb). Indeed, $\sigma_a(T+R)=\{0\}$, $\sigma_{SBF_+^-}(T+R)=\emptyset$, $\Pi^0(T+R)=\Pi_a^0(T+R)=\emptyset$. Note also that $\Pi_a(T+R)=\{0\}$.

Moreover, this example shows that the result obtained in Proposition 4.1 cannot be extended to commuting Riesz operators. Nonetheless, we have the next result.

Theorem 4.4. *Let $T\in L(X)$ and let $R\in\mathcal{R}(X)$ which commutes with T. We have:*

i) *If T satisfies property (SBb), then $T+R$ satisfies property (SBb) if and only if $\Pi_a(T+R)=\Pi^0(T+R)$.*
ii) *If T satisfies property $(SBab)$, then $T+R$ satisfies property $(SBab)$ if and only if $\Pi_a(T+R)=\Pi_a^0(T+R)$.*

Proof. i) If $T+R$ satisfies (SBb), then from Theorem 1.3 we have $\Pi^0(T+R)=\Pi_a(T+R)$. Conversely, suppose that $\Pi^0(T+R)=\Pi_a(T+R)$. Since T satisfies property (SBb) then it satisfies a-Browder's theorem. By Lemma 3.5, $T+R$ satisfies generalized a-Browder theorem. Thus $\sigma_a(T+R)\setminus\sigma_{SBF_+^-}(T+R)=\Pi_a(T+R)=\Pi^0(T+R)$. So $T+R$ satisfies property (SBb).

ii) If $T+R$ satisfies property $(SBab)$, then $\Pi_a(T+R) = \Pi_a^0(T+R)$, see [3, Theorem 2.14]. Conversely, assume that $\Pi(T+R) = \Pi_a^0(T+R)$. Since T satisfies property $(SBab)$ then it satisfies generalized a-Browder's theorem. Hence generalized a-Browder's theorem holds for $T+R$, that is: $\sigma_a(T+R) \setminus \sigma_{SBF_+^-}(T+R) = \Pi_a(T+R)$. So $T+R$ satisfies property $(SBab)$. ∎

As an application of Theorem 4.4 to the class of quasi-nilpotent operators, we give two corollaries.

Corollary 4.5. *Let $T \in L(X)$ and let $Q \in \mathcal{Q}(X)$ such that $TQ = QT$.*

i) *If T satisfies property (SBb), then the following statements are equivalent:*
   a) *$T+Q$ satisfies property (SBb),*
   b) *$\Pi_a(T+Q) = \Pi^0(T)$,*
   c) *$\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$.*

ii) *If $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$ or $\operatorname{iso}\sigma_{ub}(T) = \emptyset$, then T satisfies property (SBb) if and only if $T+Q$ satisfies property (SBb).*

Proof. i) a) ⇔ b) Since T commutes with Q, we know that $\sigma(T+Q) = \sigma(T)$ and $\sigma_a(T+Q) = \sigma_a(T)$. By [16, Corollary 8], we have $\sigma_b(T+Q) = \sigma_b(T)$. So $\Pi^0(T+Q) = \sigma(T+Q) \setminus \sigma_b(T+Q) = \sigma(T) \setminus \sigma_b(T) = \Pi^0(T)$. Hence the equivalence between statements a) and b) is a consequence of Theorem 4.4.

a) ⇔ c) If $T+Q$ satisfies property (SBb) then $\sigma_{SBF_+^-}(T+Q) = \sigma_a(T+Q) \setminus \Pi^0(T+Q) = \sigma_a(T) \setminus \Pi^0(T) = \sigma_{SBF_+^-}(T)$, since T satisfies property (SBb). Conversely, assume that $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$. Then $\sigma_a(T+Q) \setminus \sigma_{SBF_+^-}(T+Q) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T) = \Pi^0(T+Q)$. So $T+Q$ satisfies property (SBb).

ii) Case 1. $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$: assume that T satisfies property (SBb). The condition $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$ implies from [8, Proposition 2.4] that $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$. So from the assertion i), it follows that $T+Q$ satisfies property (SBb). Conversely, assume that $T+Q$ satisfies property (SBb). Since $\operatorname{iso}\sigma_{SF_+^-}(T+Q) = \emptyset$ and $T = (T+Q) - Q$, we conclude similarly.

Case 2. $\operatorname{iso}\sigma_{ub}(T) = \emptyset$: assume that T satisfies property (SBb). The condition $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ implies from [8, Corollary 2.11] that $\Pi_a(T+Q) = \Pi_a(T)$, and since T satisfies property (SBb) then $\Pi_a(T) = \Pi^0(T)$. So $\Pi_a(T+$

$Q) = \Pi^0(T)$ and hence $T+Q$ satisfies property (SBb). We obtain the proof of the converse analogously, since $\text{iso}\,\sigma_{ub}(T+Q) = \emptyset$. ∎

In the following corollary, we give a similar perturbation result for the property $(SBab)$.

Corollary 4.6. *Let $T \in L(X)$ and let $Q \in \mathcal{Q}(X)$ such that $TQ = QT$.*

i) *If T satisfies property $(SBab)$, then the following statements are equivalent:*
a) $T+Q$ satisfies property $(SBab)$,
b) $\Pi_a(T+Q) = \Pi_a^0(T)$,
c) $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$.

ii) *If $\text{iso}\,\sigma_{SF_+^-}(T) = \emptyset$ or $\text{iso}\,\sigma_{ub}(T) = \emptyset$, then T satisfies property $(SBab)$ if and only if $T+Q$ satisfies property $(SBab)$.*

Proof. Is similar to the proof of the precedent corollary. ∎

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